Exo1

1) $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \lim_{(x,y)\to(0,0)} f_1(x,y)$.

$x = r\cos\theta$ $\quad$ $(r \to 0 \Rightarrow (x,y)\to(0,0))$

$y = r\sin\theta$

$\lim_{r\to 0} \frac{r^2\cos^2\theta - r^2\sin^2\theta}{r^2\cos^2\theta + r^2\sin^2\theta} = \lim_{r\to 0} \frac{r^2(\cos^2\theta-\sin^2\theta)}{r^2}$

$= \cos^2\theta - \sin^2\theta$

la limite depend de θ donc $\lim_{(x,y)\to(0,0)} f_1(x,y)$ n'existe pas

2) $\lim_{(x,y)\to(0,0)} f_2(x,y) = \lim_{(x,y)\to(0,0)} \frac{x\sin\frac{1}{x}+y}{x+y}$

$\lim_{y\to 0}\left(\lim_{x\to 0} f_2(x,y)\right) = \lim_{y\to 0} \frac{y}{y} = 1$.

$\lim_{x\to 0}\left(\lim_{y\to 0} f_2(x,y)\right) = \lim_{x\to 0} \sin\frac{1}{x}$ n'existe pas.

On a $\lim_{y\to 0}\left(\lim_{x\to 0} f_2(x,y)\right) \neq \lim_{x\to 0}\lim_{y\to 0}\left(f_2(x,y)\right)$ d'ou.

$\lim_{(x,y)\to(0,0)} f_2(x,y)$ n'existe pas.

3) $\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2}$

$x = r\cos\theta$, $y = r\sin\theta$

$\lim_{r\to 0} \frac{r^3(\cos^3\theta-\sin^3\theta)}{r^2} = \lim_{r\to 0} r(\cos^3\theta - \sin^3\theta) = 0$

4) $\lim\limits_{(x,y)\to(0,0)} f_4(x,y) = \lim\limits_{(x,y)\to(0,0)} \frac{\sin xy}{x}$

$= \lim\limits_{(x,y)\to(0,0)} \frac{y \sin xy}{xy} = 0$

$\frac{\sin xy}{xy} \to 1$

5) $\lim\limits_{(x,y)\to(0,0)} f_5(x,y) = \lim\limits_{(x,y)\to(0,0)} \left(x^2+y^2\right)^{x^2y^2}$

$= \lim\limits_{(x,y)\to(0,0)} e^{x^2y^2 \ln\left(x^2+y^2\right)}$

$x = r\cos\theta, \quad y = r\sin\theta$

$\lim\limits_{(x,y)\to(0,0)} f_5(x,y) = \lim\limits_{r\to 0} e^{r^2\cos^2\theta\sin^2\theta \ln\left(r^2\right)} = 1$

6) $\lim\limits_{(x,y)\to(0,0)} \frac{3x^2+yx}{\sqrt{x^2+y^2}}$

$x = r\cos\theta, \quad y = r\sin\theta, \quad r\to 0$

$\lim\limits_{r\to 0} \frac{3r^2\cos^2\theta + r^2\cos\theta\sin\theta}{r} = \lim\limits_{r\to 0} r\left(3\cos^2\theta + \cos\theta\sin\theta\right)$

$= 0$

7) $\lim\limits_{(x,y)\to(0,0)} \frac{|x|+|y|}{x^2+y^2}$

$\lim\limits_{r\to 0} \frac{|r|\left(|\cos\theta|+|\sin\theta|\right)}{r^2} = +\infty$

8) $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin x^2 - \sin y^2}{x^2+y^2}$.

$y = x$, $x \to 0$, $(x,y) \to (0,0)$

$\lim\limits_{x\to 0} \dfrac{\sin x^2 - \sin x^2}{2x^2} = 0$

$y = \sqrt{2}\,x$

$\lim\limits_{x\to 0} \dfrac{\sin x^2 - \sin 2x^2}{x^2 + 2x^2} = \lim\limits_{x\to 0} \dfrac{\sin x^2 - 2\sin x^2 \cos x^2}{3x^2} =$

$\lim\limits_{x\to 0} \dfrac{\overbrace{\sin x^2}^{\to 1}(1 - 2\cos x^2)}{x^2 \times 3} = -\dfrac{1}{3}$.

donc $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin^2 x - \sin y^2}{x^2+y^2}$ n'existe pas.

9) $\lim\limits_{(x,y)\to(+\infty,+\infty)} \dfrac{x+y}{x^2+y^2}$

changement de variable

$x = \dfrac{1}{X}$, $y = \dfrac{1}{Y}$

$(X,Y) \to (0,0) \Rightarrow (x,y) \to (+\infty,+\infty)$

$\lim\limits_{(X,Y)\to(0,0)} \dfrac{\frac{1}{X} + \frac{1}{Y}}{\frac{1}{X^2} + \frac{1}{Y^2}} = \lim\limits_{(X,Y)\to(0,0)} \dfrac{(Y+X)(XY)}{X^2+Y^2}$ $\Big/ \begin{array}{l} X = r\cos\theta \\ Y = r\sin\theta \end{array}$

$= \lim\limits_{r\to 0} \dfrac{r^3(\sin\theta + \cos\theta)(\cos\theta\sin\theta)}{r^2}$

$= 0$

Left margin: $l:$; $(x,y$; $=$; $2)$; $\dfrac{\partial f}{\partial x}$; $\dfrac{\partial f}{\partial y}$

Exo2

$$f(x,y)=\begin{cases}\dfrac{x^3 y}{x^6+y^2} & \text{si } (x,y)\neq(0,0)\\ 0 & \text{sinon.}\end{cases}$$

1) Etude de la continuite de f en (0,0)

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)=0 \,?$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^6+y^2}$$

$y=x^3$.

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{x^6}{2x^6} = \frac{1}{2}.$$

$y=2x^3$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{2x^6}{5x^6} = \frac{2}{5}.$$

d'ou $\lim_{(x,y)\to(0,0)} f(x,y)$ n'escite pas par suite

f n'est pas contine en (0,0).

2) Calcul des dérivées partielles en (0,0)

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{k\to 0} \frac{f(0,k+0)-f(0,0)}{k} = 0$$

d'ou les dérivées en ...