TD #5

Exercice 3

h $f(x) = \cos x^2 \quad \forall x \in \mathbb{R}$

$$\cos = 1 - \frac{{}^2}{2!} + \frac{{}^4}{4!} - \frac{{}^6}{6!} + \frac{{}^8}{8!} - \dots + (-1)^n \frac{{}^{2n}}{(2n)!} + \dots$$

si on substitue $X = 2x^2$ on obtient

$$\cos x^2 = 1 - \frac{(x^2)^2}{2!} + \frac{(x^2)^4}{4!} - \frac{(x^2)^6}{6!} + \frac{(x^2)^8}{8!} - \dots + (-1)^n \frac{(x^2)^{2n}}{(2n)!} + \dots$$

$$\cos x^2 = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \frac{x^{16}}{8!} - \dots + (-1)^n \frac{x^{4n}}{(2n)!} + \dots \quad \forall x \in \mathbb{R}$$

i $f(x) = \ln(1 - x^3)$

$$\ln(1 - X) = -X - \frac{X^2}{2} - \frac{X^3}{3} - \frac{X^4}{4} - \dots - \frac{X^n}{n} - \dots \qquad -1 \leq X < 1$$

$-1 \leq x^3 < 1.$

on Substitue $X = x^3$ $\qquad -1 \leq x < 1.$

$$\ln(1 - x^3) = -x^3 - \frac{(x^3)^2}{2} - \frac{(x^3)^3}{3} - \frac{(x^3)^4}{4} - \dots - \frac{(x^3)^n}{n} - \dots$$

$$\ln(1 - x^3) = -x^3 - \frac{x^6}{2} - \frac{x^9}{3} - \frac{x^{12}}{4} - \dots - \frac{x^{3n}}{n} - \dots \qquad -1 \leq x < 1$$

j $f(x) = e^{1-x} = e \cdot e^{-x}$

$$e^X = 1 + X + \frac{X^2}{2!} + \frac{X^3}{3!} + \frac{X^4}{4!} + \dots + \frac{X^n}{n!} + \dots \qquad X \in \mathbb{R}$$

$X = 1 - x$

$$e^{1-x} = 1 + (1-x) + \frac{(1-x)^2}{2!} + \frac{(1-x)^3}{3!} + \frac{(1-x)^4}{4!} + \dots + \frac{(1-x)^n}{n!} \qquad x \in \mathbb{R}$$

$$= 1 + 1 - x + \frac{1 - 2x + x^2}{2!} + \frac{1 - 3x + 3x^2 + x^3}{3!} + \frac{1 - 4x + 6x^2 - 4x^3 + x^4}{4!} + \frac{1 - 5x + 10x^2 - 10x^3 + 5x^4 - x^5}{5!}$$

$$+ \frac{C_n^0 - C_n^1 x + C_n^2 x^2 - C_n^3 x^3 + \dots + (-1)^n C_n^n x^n}{n!}$$

$$= 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!} + \dots + x\left(-1 - \frac{2}{2!} - \frac{3}{3!} - \frac{4}{4!} \dots - \frac{n}{n!} \dots\right) + x^2\left(\frac{1}{2!} + \frac{3}{3!} + \frac{6}{4!} + \frac{10}{5!} + \dots\right)$$

$$= e - ex + \frac{ex^2}{2!} - \frac{e}{3!}x^3 + \frac{e}{4!}x^4 \dots + (-1)\frac{e}{n!}x^n$$

$$= e\left(1 - x + \frac{x^2}{2!}\right.$$

k) $\tanh x = \dfrac{\operatorname{sh} x}{\operatorname{ch} x}$

$\tanh x \cdot \operatorname{ch} x = \operatorname{sh} x$

$$(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots)\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots + \frac{x^{2n}}{(2n)!} + \dots\right) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots + \frac{x^{2n+1}}{(2n+1)!} + \dots$$

$$a_0 + a_1 x + (a_0 + a_2)x^2 + \left(\frac{a_1}{2} + a_3\right)x^3 + \dots = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots + \frac{x^{2n+1}}{(2n+1)!}$$

$a_0 = 0$

$a_1 = 1$

$a_0 + a_2 = 0 \implies a_2 = -a_0 = 0$

$\dfrac{a_1}{2} + a_3 = \dfrac{1}{6} \implies a_3 = \dfrac{1}{6} - \dfrac{1}{2} = -\dfrac{1}{3}$

$$\tanh x = x - \frac{x^3}{3} + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \dots$$

l) $\tan x = \dfrac{\sin x}{\cos x}$ $\quad x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$

comme precedement $\quad \tan x = x + \dfrac{x^3}{3} + \dfrac{2x^5}{15} + \dfrac{17}{315}x^7 + \dots$

m) $f(x) = \dfrac{e^x}{\cos x}$ $\quad \begin{array}{l} \nearrow x \in R \\ \searrow x \in R \neq \frac{\pi}{2} + k\pi \end{array}$

même procedure $\quad \dfrac{e^x}{\cos x} = 1 + x + x^2 + \dfrac{2x^3}{3} + \dfrac{x^4}{2} + \dfrac{3x^5}{10} + \dots$

n) $\sec x = \dfrac{1}{\cos x} = \dfrac{1}{1 - (1 - \cos x)}$

$$\sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \frac{277x^8}{8064} + \dots$$

$$= \sum_{n=0}^{\infty} (-1)^n \operatorname{euler}(2n) \frac{x^{2n}}{(2n)!}$$

Exercice 4

(a) $f(x) = \cos x$ avec $f(x) = \frac{d}{dx}(\sin x)$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + \frac{x^{2n+1}}{(2n+1)!}(-1)^n + \dots$$

dérivée $\downarrow$ $\qquad\downarrow$ dérivation terme à terme

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + \frac{x^{2n}}{(2n)!}(-1)^n + \dots$$

(b) $f(x) = \frac{1}{2\sqrt{1+x}}$ avec $\frac{d}{dx}\left[(1+x)^{1/2}\right] = f(x)$

$$= (1+x)^{1/2} = 1 + \frac{1}{2}x + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}x^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}x^3 + \dots + \frac{\frac{1}{2}(\frac{1}{2}-1)\dots(\frac{1}{2}-n+1)}{n!}x^n + \dots$$

dérivé $\downarrow$ $\qquad\downarrow$ différentiation terme par terme

$$\frac{1}{2\sqrt{1+x}} = \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2}-1\right)x + \frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\frac{x^2}{2!} + \dots + \frac{1}{2}\left(\frac{1}{2}-1\right)\dots\left(\frac{1}{2}-n+1\right)\frac{x^{n-1}}{(n-1)!} + \dots$$

$$+ \dots \quad \frac{1}{2}\left(\frac{1}{2}-1\right)\dots\left(\frac{1}{2}-n\right)\frac{x^n}{n!} + \dots$$

$$\frac{1}{2\sqrt{1+x}} = \frac{1}{2} + \frac{x}{4} - \frac{x^2}{16} + \frac{x^3}{32} - \frac{5x^4}{256} + \frac{7x^5}{512} + \dots$$

Exercice 5

(a) $f(x) = \tanh^{-1}x$ avec $\frac{d}{dx}(\tanh^{-1}x) = \frac{1}{1-x^2}$

$$\frac{1}{1-x^2} = \frac{1}{1-X} = 1 + X + X^2 + X^3 + \dots + X^n + \dots$$

$\downarrow$ $X = x^2$

$$= 1 + x^2 + x^4 + x^6 + \dots + x^{2n} + \dots$$

$$\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + \dots + x^{2n} + \dots$$

intégration $\downarrow$ $\qquad\downarrow$ terme-wise intégration

$$\tanh^{-1}x = \text{cte} + x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \dots + \frac{x^{2n+1}}{2n+1} + \dots \qquad \tanh^{-1} 0 = 0 \Rightarrow \text{Cte} = 0$$

$$\boxed{\tanh^{-1}x = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \dots + \frac{x^{2n+1}}{2n+1} + \dots}$$

b $f(x) = \int_0^x e^{-t^2}\,dt$

$$e^{-t^2} = 1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \frac{t^8}{4!} - \dots + \frac{(-1)^n t^{2n}}{n!} + \dots$$

intégration terme à terme

$$\int e^{-t^2}\,dt = cte + t - \frac{t^3}{3} + \frac{t^5}{5\cdot 2!} - \frac{t^7}{7\cdot 3!} + \frac{t^9}{9\cdot 4!} + \dots + \frac{(-1)^n\, t^{2n+1}}{(2n+1)\cdot n!} + \dots$$

$$\int_0^x e^{-t^2}\,dt = \left(cte + x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \frac{x^9}{9\cdot 4!} + \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)\cdot n!} \right) - \left(cte + 0 + 0 + \dots \right)$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \frac{x^9}{9\cdot 4!} + \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)\cdot n!} + \dots$$

c $f(x) = sh^{-1} x$ avec $\frac{d}{dx}(sh^{-1}x) = \frac{1}{\sqrt{1+x^2}}$

$$\frac{1}{\sqrt{1+x^2}} = (1+x^2)^{-1/2} = (1+X)^{-1/2} \quad \text{avec } X = x^2 \qquad -1 < X < 1,\ -1 < x < 1$$

$$\frac{1}{\sqrt{1+x^2}} = 1 + kX + \frac{k(k-1)}{2!}X^2 + \frac{k(k-1)(k-2)}{3!}X^3 + \dots + \frac{k(k-1)(\dots)(k-n+1)}{n!}X^n + \dots$$

$$\frac{1}{\sqrt{1+x^2}} = 1 - \frac{1}{2}x^2 + \frac{-1/2(-1/2-1)}{2!}x^4 + \frac{-1/2(1/2-1)(-1/2-2)}{3!}x^6 + \dots + \frac{-1/2(-1/2-1)\dots(-1/2-n+1)}{n!}x^{2n} + \dots$$

$$\frac{1}{\sqrt{1+x^2}} = 1 - \frac{x^2}{2} + \frac{3x^4}{8} - \frac{5x^6}{16} + \frac{35x^8}{128} - \frac{63x^{10}}{256} + \dots + \frac{-1/2(-1/2-1)\dots(-1/2-n+1)\,x^{2n+1}}{(2n+1)\cdot n!} + \dots$$

intégration

$$sh^{-1}x = cte + x - \frac{x^3}{6} + \frac{3x^5}{40} - \frac{5x^7}{112} + \dots \qquad sh^{-1}0 = 0 \Rightarrow cte = 0$$

$$sh^{-1}x = x - \frac{x^3}{6} + \frac{3x^5}{40} - \frac{5x^7}{112} + \dots$$

d $f(x) = \sec^2 x = \frac{1}{\cos^2 x}$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$\tan x =$

c) $f(x) = \tan^{-1} x \qquad \frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$

$$\frac{1}{1+x^2} = \frac{1}{1-\underbrace{(-x^2)}_{X}} = 1 + X + X^2 + X^3 + X^4 + \dots + X^n =$$

$$= 1 - x^2 + x^4 - x^6 + x^8 - \dots + x^{2n}(-1)^n + \dots$$

$\downarrow$ integration $\qquad\qquad \downarrow$ integration

$$\tan^{-1} x = cte + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} + \dots + (-1)^n \frac{x^{2n+1}}{2n+1} + \dots$$

$cte = 0$ car $\tan^{-1} 0 = 0$.