TD#1 Suites et Series
Intro

Exercice 2

ⓐ $1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \frac{1}{25}, \ldots$ $\qquad a_n = (-1)^{n+1}\frac{1}{n^2} \qquad n=1,\cdots,\infty$

ⓑ $0, 1, 1, 2, 2, 3, 3, 4, \ldots$

$n \to 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8$

$$a_n = \begin{cases} \frac{n}{2} & \text{si } n \text{ pair} \\ \frac{n-1}{2} & \text{si } n \text{ impair} \end{cases} \qquad n=1,\cdots,\infty$$

ⓒ $2, 6, 10, 14, 18$

$n\quad 0\ 1\ 2\ 3\ 4$

$a_n = 2 + 4n \qquad n = 0, 1, 2, \ldots$

ou

$a_0 = 2, \quad a_{n+1} = a_n + 4 \qquad n = 1, 2, \cdots, \infty$

Exercice 3

ⓐ $a_n = \frac{3n+1}{n+1}$

n	0	1	2	3	4
a_n	0	2	$2\frac{1}{3}$	$2\frac{1}{2}$	2,6

il semble que La suite est monotone croissante

$a_{n+1} > a_n$?

$$\frac{3(n+1)+1}{(n+1)+1} > \frac{3n+1}{n+1} \iff \frac{3n+4}{n+2} > \frac{3n+1}{n+1}$$

$$\iff (3n+4)(n+1) > (3n+1)(n+2)$$

$$\iff 3n^2 + 7n + 4 > 3n^2 + 7n + 2 \iff 4 > 2 \quad \text{toujours vrai}$$

donc la suite est monotone croissante.

ⓑ $a_n = \frac{(2n+3)!}{(n+1)!}$

$a_n = (n+2)(n+3)\cdots(2n+1)(2n+2)(2n+3)$

$a_{n+1} = (n+3)(n+4)\cdots\cdots(2n+3)(2n+4)(2n+5)$

$\frac{a_{n+1}}{a_n} = \frac{(2n+4)(2n+5)}{n+2} > 1$ donc la suite est monotone croissante

(c) $a_n = \dfrac{2^n 3^n}{n!}$

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}3^{n+1}}{(n+1)!} \times \frac{n!}{2^n 3^n} = \frac{2\cdot 3}{n+1} \leqslant 1 \text{ si } n \geqslant 5$$

donc la suite est monotone decroissante pour $n \geqslant 5$

(d) $a_n = 2 - \dfrac{2}{n} - \dfrac{1}{2^n}$

$$a_{n+1} - a_n = 2 - \frac{2}{n+1} - \frac{1}{2^n} - \left(2 - \frac{2}{n} - \frac{1}{2^{n+1}}\right)$$

$$= \cancel{2} - \cancel{2} + 2\left(\frac{1}{n} - \frac{1}{n+1}\right) + \frac{1}{2^n}\left(\frac{1}{2} - 1\right)$$

$$= \frac{2}{n(n+1)} - \frac{1}{2^{n+1}} = \frac{2^{n+2} - n(n+1)}{n(n+1)}$$

$a_{n+1} - a_n > 0$ si $2^{n+2} - n(n+1) > 0$

$2^{n+2} > n(n+1)$ vrai mais pour quel n? $\underline{\underline{n > N}}$

$2^{x+2} > x(x+1) \sim x^2$ qd $x \to \infty$

vrai car $\displaystyle\lim_{x\to\infty} \frac{e^x}{x^N} = 0 \quad \forall \underline{\underline{N}}$

Exercice 4

$x_n = x_{n-1} + \cos x_{n-1}$

n	0	1	2	3	4	5
x_n	0	1	1.540302	1.570791	1.570796	$1.570796 \approx \frac{\pi}{2}$

si la suite converge, alors $x_n = x_{n-1} = L$

en substituant dans la formule d'iteration, on trouve

$$L = L + \cos L \Rightarrow \cos L = 0 \Rightarrow L = \pi/2 \,!$$

pourquoi l'iteration converge? $\longrightarrow$ Cours Analyse Numerique methodes iteratives!

Exercice 5

① $a_n = \dfrac{1-2n}{1+2n}$

$f(x) = \dfrac{1-2x}{1+2x}$ continue pour $x \in \mathbb{R}^+$

$\lim\limits_{x\to\infty} f(x) = \dfrac{-2}{2} = -1$

$a_n = f(n)$ donc $\boxed{\lim\limits_{n\to\infty} a_n = -1}$ la suite converge

② $a_n = \dfrac{n!}{n^n}$

On compare avec la suite $b_n = \dfrac{1}{n}$

$c_n = \dfrac{a_n}{b_n}$ $\lim\limits_{n\to\infty} c_n = ?$ on veut une constante

$c_n = \dfrac{n!}{n^n} \times n = \dfrac{n!}{n^{n-1}}$ de même $c_{n+1} = \dfrac{(n+1)!}{(n+1)^{n+1-1}} = \dfrac{(n+1)!}{(n+1)^n}$

$$\frac{c_{n+1}}{c_n} = \frac{(n+1)!}{(n+1)^n} \times \frac{n^{n-1}}{n!} = \frac{n+1}{(n+1)^n} \cdot n^{n-1} = \frac{n^{n-1}}{(n+1)^{n-1}} = \left(\frac{n}{n+1}\right)^{n-1}$$

$$= \left(\frac{n+1-1}{n+1}\right)^{n-1} = \left(1 - \frac{1}{n+1}\right)^{n-1} = \frac{\left(1 - 1/_{n+1}\right)^{n+1}}{\left(1 - 1/_{n+1}\right)^2}$$

$\lim\limits_{n\to\infty} \left(1 + \dfrac{x}{n}\right)^n = e^x$ donc

$$\lim_{n\to\infty} \frac{c_{n+1}}{c_n} = \lim_{n\to\infty} \frac{\left(1 - \frac{1}{n+1}\right)^{n+1}}{\left(1 - 1/_{n+1}\right)^2} = \frac{e^{-1}}{1} = \frac{1}{e} < 1$$

donc $\exists\, N_0 \in \mathbb{N}$ tel que $\forall n > N_0$ $c_{n+1} < c_n$

donc la suite c_n est

- décroissante monotone
- minorée par zero (>0)

$\Big\}\Rightarrow$ c_n Converge $\longrightarrow L > 0$

$\lim\limits_{n\to\infty} c_n = \lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = L \longrightarrow a_n \sim b_n$ donc

puisque $\sum b_n$ diverge donc $\sum a_n$ diverge !

③ $a_n = \sqrt[n]{n}$

il est connu que $\lim\limits_{x\to\infty} x^{1/x} = \lim\limits_{x\to\infty} e^{\frac{1}{x}\ln x} = e^0 = 1$

donc $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \sqrt[n]{n} = 1$

la suite converge vers $L = 1$

④ $a_n = \dfrac{2n+1}{1-3\sqrt{n}}$

$f(x) = \dfrac{2x+1}{1-3\sqrt{x}}$ $\quad \lim\limits_{x\to\infty} f(x) = \dfrac{2x}{-3\sqrt{x}} = -\infty$

donc $\lim\limits_{n\to\infty} a_n = -\infty$ la suite diverge

⑤ $a_n = \left(1+\dfrac{7}{n}\right)^n$

$\lim\limits_{n\to\infty} \left(1+\dfrac{7}{n}\right)^n = e^7$ donc la suite converge vers $l = e^7$

⑥ $a_n = n - \sqrt{n^2-n}$ $\quad n\to\infty \Rightarrow a_n = \infty - \infty$ forme indeterminée

$$a_n = \frac{(n-\sqrt{n^2-n})(n+\sqrt{n^2-n})}{n+\sqrt{n^2-n}} = \frac{n^2-(n^2-n)}{n+\sqrt{n^2-n}} = \frac{n}{n+\sqrt{n^2-n}}$$

$= \dfrac{1}{1+\sqrt{1-\frac{1}{n}}}$ donc $\lim\limits_{n\to\infty} a_n = \dfrac{1}{2}$ la suite converge vers $\frac{1}{2}$

⑦ $a_n = \dfrac{\sin^2 n}{2^n}$

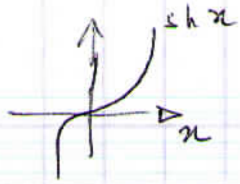
$0 \leqslant a_n \leqslant \dfrac{1}{2^n} = b_n$ $\quad \lim\limits_{n\to\infty} b_n = 0 \Rightarrow \lim\limits_{n\to\infty} a_n = 0$

⑧ $a_n = \dfrac{\ln(n+1)}{\sqrt{n}}$

$a_n = \dfrac{\ln(n+1)}{\sqrt{n}} \leqslant \dfrac{\ln n}{\sqrt{n}} = \dfrac{\ln(n)}{n^{1/2}} = b_n$

$\lim\limits_{x\to\infty} \dfrac{\ln x}{x^\alpha} = 0$ $\quad \forall\, \alpha > 0$ donc $\underset{\text{aussi}}{\alpha = \frac{1}{2}}$ $\Rightarrow \lim\limits_{n\to\infty} a_n = 0$

(9) $a_n = \operatorname{sh}(\ln n)$



$\lim\limits_{n\to\infty} a_n = \infty$ diverge

(10) $a_n = \underbrace{(-1)^n}_{\text{diverge}} \underbrace{\left(1 - \frac{1}{n}\right)}_{\text{Converge vers } 1}$

la suite $-1, +1, -1$ lorsque $n \to \infty$ elle diverge.

(11) $a_n = \overbrace{\left(\frac{n+1}{2n}\right)}^{b_n} \overbrace{\left(1 - \frac{1}{n}\right)}^{c_n}$

$\left(\frac{n+1}{2n}\right) \to \frac{1}{2}$, $\left(1 - \frac{1}{n}\right) \to 1$

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n \times \lim\limits_{n\to\infty} c_n = \frac{1}{2} \times 1 = \frac{1}{2}$

la suite Converge vers $L = \frac{1}{2}$

(12) $a_n = \overbrace{\left(\frac{1}{3}\right)^n}^{b_n} + \overbrace{\frac{1}{\sqrt{2^n}}}^{c_n}$ $\qquad a_n = b_n + c_n$

$\left(\frac{1}{3}\right)^n \to 0$, $\frac{1}{\sqrt{2^n}} \to 0$

$\lim\limits_{n\to\infty} b_n = 0 \qquad \lim\limits_{n\to\infty} c_n = 0$

donc $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n + \lim\limits_{n\to\infty} c_n = 0$

la suite converge vers zero.

Exercice 6

(1) $1 + 3 + 5 + 7 + 9 + \cdots$

$S_1 = 1 \qquad S_2 = 1 + 3 = 4 \qquad S_3 = 1 + 3 + 5 = 9$

$S_4 = S_3 + 7 = 9 + 7 = 16$

(3) $1 - 1 + 1 - 1 + 1 - 1 + \cdots$

$S_1 = 1 \qquad S_2 = 1 - 1 = 0 \qquad S_3 = S_2 + 1 = 1 \qquad S_4 = S_3 - 1 = 0$

Exercice 7

① $2+\frac{2}{3}+\frac{2}{9}+\frac{2}{27}+\dots+\frac{2}{3^{n-1}}+\dots$

$$2\left(1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\dots+\frac{1}{3^{n-1}}+\frac{1}{3^n}+\dots\right)$$

$S_0 = 1 \qquad S_1 = 1+\frac{1}{3} \quad \dots$

$$S_n = 1+\frac{1}{3}+\frac{1}{9}+\dots+\frac{1}{3^n}$$

$$1/3\, S_n = \frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\dots+\frac{1}{3^{n+1}}$$

$$S_n - \frac{1}{3}S_n = 1-\frac{1}{3^{n+1}}$$

$$S_n\left(1-1/3\right) = 1-1/3^{n+1}$$

$$S_n = \frac{1-\left(\frac{1}{3}\right)^{n+1}}{1-1/3}$$

$$S_n = \frac{3-\frac{1}{3^n}}{2}$$

$$S_n = 1+r+r^2+\dots+r^n$$

$$r S_n = r+r^2+\dots+r^n+r^{n+1}$$

$$S_n - rS_n = 1-r^{n+1}$$

$$S_n(1-r) = 1-r^{n+1}$$

$$S_n = \frac{1-r^{n+1}}{1-r}$$

② $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\dots+(-1)^n\frac{1}{2^{n-1}}+(-1)^{n+1}\frac{1}{2^n}+\dots$

$S_n = 1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\dots+(-1)^{n+1}\frac{1}{2^n}$ suite géométrique $r=-\frac{1}{2}$

$$S_n = \frac{1-\left(-1/2\right)^{n+1}}{1-\left(-1/2\right)} = \frac{1+\frac{(-1)^n}{2^{n+1}}}{3/2} = \frac{2}{3}\left(1+\frac{(-1)^n}{2^{n+1}}\right)$$

③ $\frac{1}{2\cdot3}+\frac{1}{3\cdot4}+\frac{1}{4\cdot5}+\dots+\frac{1}{(n+1)(n+2)}+\dots$

$$S_n = \underset{\substack{\downarrow\\ n=1}}{\frac{1}{2\cdot3}}+\frac{1}{3\cdot4}+\frac{1}{4{,}5}+\dots+\underset{\substack{\downarrow\\ n=n}}{\frac{1}{(n+1)(n+2)}}$$

$$\frac{1}{(n+1)(n+2)} = \frac{1}{n+1}-\frac{1}{n+2}$$

$$S_n = \left(\frac{1}{2}-\cancel{\frac{1}{3}}\right)+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}\right)+\left(\cancel{\frac{1}{4}}-\cancel{\frac{1}{5}}\right)+\dots+\left(\cancel{\frac{1}{n}}-\cancel{\frac{1}{n+1}}\right)+\left(\cancel{\frac{1}{n+1}}-\frac{1}{n+2}\right)$$

$$S_n = \frac{1}{2}-\frac{1}{n+2} = \frac{n}{2(n+2)}$$

Exercice 8

ⓐ $\displaystyle\sum_{n=2}^{\infty} \frac{1}{4^n} = \frac{1}{4^2} + \frac{1}{4^3} + \frac{1}{4^4} + \dots$

$$= \frac{1}{4^2}\left(1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \dots + \frac{1}{4^n} + \dots\right)$$ Serie geomet. $r = 1/4$

$$= \frac{1}{4^2}\left(\frac{1}{1 - 1/4}\right) = \frac{1}{12}$$

ⓑ $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{5}{3^n} = 5 - \frac{5}{3} + \frac{5}{3^2} - \dots$ $\quad \left(-\frac{1}{3}\right)^n$

$$= 5\left(1 - \frac{1}{3} + \frac{1}{3^2} - \frac{1}{3^3} + \dots + \frac{(-1)^n}{3^n} + \dots\right)$$

$$= 5 \; \frac{1}{1 - (-1/3)} = 5 \times \frac{3}{4} = \frac{15}{4}$$

ⓒ $\displaystyle\sum_{n=0}^{\infty} \left(\frac{7}{2^n} - \frac{2^{n+1}}{3^n}\right) = 7 - \frac{2}{1} + \frac{7}{2} - \frac{2^2}{3} + \frac{7}{2^2} - \frac{2^3}{3^2} + \dots$

$$\sum_{n=0}^{\infty} \frac{7}{2^n} - \sum_{n=0}^{\infty} \frac{2^{n+1}}{3^n} = 7 \sum_{n=0}^{\infty} \frac{1}{2^n} - 2 \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$$

$$= 7 \; \frac{1}{1 - \frac{1}{2}} - 2 \; \frac{1}{1 - 2/3} = 14 - 6 = \boxed{8}$$

ⓓ $\displaystyle\sum_{n=1}^{\infty} \frac{4}{(4n-3)(4n+1)}$

$$\frac{4}{(4n-3)(4n+1)} = \frac{-1}{4n-3} + \frac{1}{4n+1} = \frac{1}{4n+1} - \frac{1}{4n-3}$$

$$S_n = \frac{4}{1 \cdot 5} + \frac{4}{5 \cdot 9} + \frac{4}{9 \cdot 13} + \frac{4}{13 \cdot 17} + \dots + \frac{4}{(4n+1)(4n-3)}$$

$$= \cancel{\frac{1}{5}} - \frac{1}{1} + \cancel{\frac{1}{9}} - \cancel{\frac{1}{5}} + \cancel{\frac{1}{13}} - \cancel{\frac{1}{9}} + \frac{1}{17} - \cancel{\frac{1}{13}} + \dots + \cancel{\frac{1}{4n-3}} - \cancel{\frac{1}{4n-7}} + \frac{1}{4n+1} - \cancel{\frac{1}{4n-3}}$$

$$\cancel{\frac{1}{4n-7}} - \cancel{\frac{1}{4n-11}}$$

$$S_n = -1 + \frac{1}{4n+1}$$

$$\lim_{n \to \infty} S_n = -1 \Rightarrow \sum_{n=1}^{\infty} \frac{4}{(4n-3)(4n+1)} = -1$$

Exercice 9

ⓐ $\sum_{n=0}^{\infty} (-1)^n x^{2n} = \sum_{x=0}^{\infty} (-x^2)^n$

Converge si $|-x^2| < 1$ cad $x^2 < 1$

cad $-1 < x < 1$

ⓑ $\sum_{n=0}^{\infty} 3\left(\frac{x-1}{2}\right)^n = 3 \sum_{n=0}^{\infty} \left(\frac{x-1}{2}\right)^n$

Converge si $\left|\frac{x-1}{2}\right| < 1$

$-1 < \frac{x-1}{2} < 1$ $\quad -2 < x-1 < 2$

$-1 < x < 3$

Exercice 10

$n = 1 \quad m = -1$

$n = m+2 \quad m = n-2$

ⓐ $\sum_{n=1}^{\infty} \frac{4}{(4n-3)(4n+1)} = \sum_{m=-1}^{\infty} \frac{4}{(4(m+2)-3)(4(m+2)+1)} = \sum_{m=-1}^{\infty} \frac{4}{(4m+5)(4m+9)}$

$n = m+2$

Exercice 11

ⓐ $0.\overline{17} = 0.17171717\ldots = \frac{17}{100} + \frac{17}{10^4} + \frac{17}{10^6} + \frac{17}{10^8} + \ldots$

$= \frac{17}{100}\left(1 + \frac{1}{10^2} + \frac{1}{10^4} + \frac{1}{10^6} + \ldots + \frac{1}{(10^2)^n} + \ldots\right)$

$= \frac{17}{100} \frac{1}{1 - 1/100} = \frac{17}{100} \frac{1}{0.99} = \frac{17}{\cancel{100}} \times \frac{\cancel{100}}{99} = \frac{17}{99}$

ⓑ $0.24\overline{5} = 0.24 + \frac{5}{10^3} + \frac{5}{10^4} + \frac{5}{10^5} + \ldots + \frac{5}{10^n} + \ldots = 0.24 + \frac{5}{10^3}\left(1 + \frac{1}{10} + \frac{1}{10^2} + \ldots + \frac{1}{10^n} + \ldots\right)$

$= 0.24 + \frac{5}{10^3} \cdot \frac{1}{1 - \frac{1}{10}} = \frac{24}{100} + \frac{5}{10^3} \cdot \frac{10}{9} = \frac{221}{900}$