Université Amar Telidji de Laghouat
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Antennes et Propagation/Électromagnétisme fondamentale

Résumés sur les opérateurs différentiels dans les différents systèmes de coordonnées

Coordonnées rectangulaires

$$\nabla\psi = \hat{x}\frac{\partial\psi}{\partial x} + \hat{y}\frac{\partial\psi}{\partial y} + \hat{z}\frac{\partial\psi}{\partial z}$$

$$\nabla\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla\times\vec{A} = \hat{x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\nabla\cdot\nabla\psi = \nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}$$

$$\nabla^2\vec{A} = \hat{x}\,\nabla^2 A_x + \hat{y}\,\nabla^2 A_y + \hat{z}\,\nabla^2 A_z$$

Coordonnées cylindriques

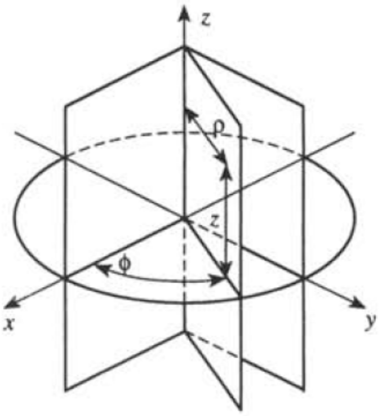
$$\nabla\psi = \hat{\rho}\frac{\partial\psi}{\partial\rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial\psi}{\partial\phi} + \hat{z}\frac{\partial\psi}{\partial z}$$

$$\nabla\cdot\vec{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla\times\vec{A} = \hat{\rho}\left(\frac{1}{\rho}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{\phi}\left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right) + \hat{z}\left(\frac{1}{\rho}\frac{\partial(\rho A_\phi)}{\partial\rho} - \frac{1}{\rho}\frac{\partial A_\rho}{\partial\phi}\right)$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

$$\vec{\nabla}^2\vec{A} = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \vec{\nabla}\times(\vec{\nabla}\times\vec{A})$$



Système de coordonnées cylindriques

Coordonnées sphériques

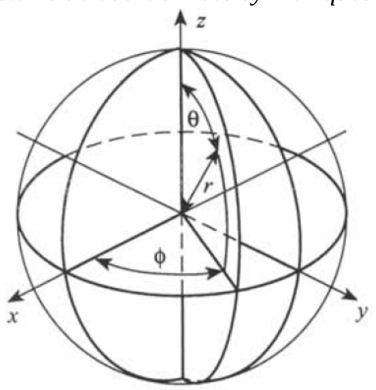
$$\nabla\psi = \hat{r}\frac{\partial\psi}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial\psi}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}$$

$$\nabla\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$\nabla\times\vec{A} = \hat{r}\,\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(A_\phi\sin\theta) - \frac{\partial A_\theta}{\partial\phi}\right) + \hat{\theta}\,\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(rA_\phi)\right) + \hat{\phi}\,\frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right)$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

$$\vec{\nabla}^2\vec{A} = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \vec{\nabla}\times(\vec{\nabla}\times\vec{A})$$



Système de coordonnées sphérique

$\oint_C \vec{A}\cdot\vec{dl} = \iint_S (\vec{\nabla}\times\vec{A})\cdot\vec{ds}$	Théorème de Stokes
$\iint_S \vec{A}\cdot\vec{ds} = \iiint_V (\vec{\nabla}\cdot\vec{A})dv$	Théorème de la divergence
$\oiint_S (\hat{n}\times\vec{A})\vec{ds} = \iiint_V (\vec{\nabla}\times\vec{A})dv$	
$\oiint_S \psi\vec{ds} = \iiint_V \vec{\nabla}\psi dv$	
$\oint_C \psi\vec{dl} = \iint_S \hat{n}\times\vec{\nabla}\psi\,\vec{ds}$	
$4\pi\vec{A} = -\vec{\nabla}\iiint\frac{\vec{\nabla}\cdot\vec{A}}{\lvert\vec{r}-\vec{r}'\rvert}d\tau' + \vec{\nabla}\times\iiint\frac{\vec{\nabla}\times\vec{A}}{\lvert\vec{r}-\vec{r}'\rvert}d\tau'$	Identité de Helmholtz

Transformation entre les différents systèmes

$$x = \rho\cos\phi = r\sin\theta\cos\phi$$
$$y = \rho\sin\phi = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$
$$\rho = \sqrt{x^2+y^2} = r\sin\theta$$
$$\phi = \tan^{-1}\frac{y}{x} = Arctg\frac{y}{x}$$
$$r = \sqrt{x^2+y^2+z^2} = \sqrt{\rho^2+z^2}$$
$$\theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z} = \tan^{-1}\frac{\rho}{z}$$

Transformation des composantes d'un vecteur d'un système à un autre

$$A_x = A_\rho\cos\phi - A_\phi\sin\phi = A_r\sin\theta\cos\phi + A_\theta\cos\theta\cos\phi - A_\phi\sin\phi$$
$$A_y = A_\rho\sin\phi + A_\phi\cos\phi = A_r\sin\theta\sin\phi + A_\theta\cos\theta\sin\phi + A_\phi\cos\phi$$
$$A_z = A_r\cos\theta - A_\theta\sin\theta$$
$$A_\rho = A_x\cos\phi + A_y\sin\phi = A_r\sin\theta + A_\theta\cos\theta$$
$$A_\phi = -A_x\sin\phi + A_y\cos\phi$$
$$A_r = A_x\sin\theta\cos\phi + A_y\sin\theta\sin\phi + A_z\cos\theta = A_\rho\sin\theta + A_z\cos\theta$$
$$A_\theta = A_x\cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta = A_\rho\cos\theta - A_z\sin\theta$$

Les éléments de volumes

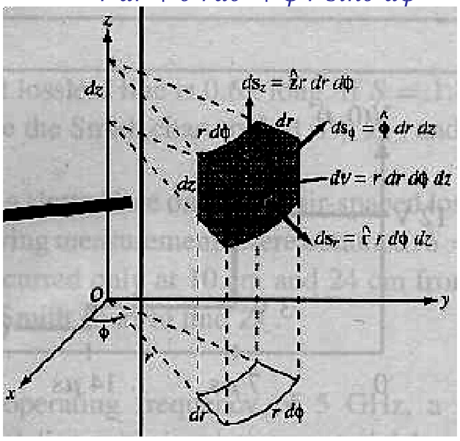
$$d\tau = dx\,dy\,dz = \rho\,d\rho\,d\phi\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$$

Les éléments de vecteurs de surfaces

$$\overrightarrow{ds} = \hat{x}\,dydz + \hat{y}\,dxdz + \hat{z}\,dxdy$$
$$= \hat{\rho}\,\rho\,d\phi dz + \hat{\phi}\,d\rho dz + \hat{z}\,\rho\,d\rho d\phi$$
$$= \hat{r}\,r^2\sin\theta\;d\theta\,d\phi + \hat{\theta}\,r\sin\theta\,drd\phi + \hat{\phi}\,rdr\,d\theta$$

Les éléments de longueur de vecteur:

$$\overrightarrow{dl} = \hat{x}\,dx + \hat{y}\,dy + \hat{z}\,dz$$
$$= \hat{\rho}\,d\rho + \hat{\phi}\,\rho d\phi + \hat{z}\,dz$$
$$= \hat{r}\,dr + \hat{\theta}\,rd\theta + \hat{\phi}\,r\sin\theta\,d\phi$$



Identités sur les vecteurs

$\vec{A}\cdot\vec{A} = \lvert\vec{A}\rvert^2$
$\vec{A}\cdot\vec{A}^* = \lvert\vec{A}\rvert^2$
$\vec{A}+\vec{B} = \vec{B}+\vec{A}$
$\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$
$\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$
$(\vec{A}+\vec{B})\cdot\vec{C} = \vec{A}\cdot\vec{C}+\vec{B}\cdot\vec{C}$
$(\vec{A}+\vec{B})\times\vec{C} = \vec{A}\times\vec{C}+\vec{B}\times\vec{C}$
$\vec{A}\times\vec{B}\times\vec{C} = (\vec{A}\cdot\vec{C})\vec{B}-= (\vec{A}\cdot\vec{B})\vec{C}$
$(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) = \vec{A}\cdot\vec{B}\times(\vec{C}\times\vec{D}) = \vec{A}\cdot(\vec{B}\cdot\vec{D}\,\vec{C}-\vec{B}\cdot\vec{C}\,\vec{D}) = (\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D})-(\vec{A}\cdot\vec{D})(\vec{B}\cdot\vec{C})$
$(\vec{A}\times\vec{B})\times(\vec{C}\times\vec{D}) = (\vec{A}\times\vec{B}\cdot\vec{D})\vec{C} - (\vec{A}\times\vec{B}\cdot\vec{C})\vec{D}$

$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0$
$\vec{\nabla}\times\vec{\nabla}\psi = 0$
$\vec{\nabla}(\phi+\psi) = \vec{\nabla}\phi+\vec{\nabla}\psi$
$\vec{\nabla}(\phi\psi) = \phi\vec{\nabla}\psi+\psi\vec{\nabla}\phi$
$\vec{\nabla}\cdot(\vec{A}+\vec{B}) = \vec{\nabla}\cdot\vec{A}+\vec{\nabla}\cdot\vec{B}$
$\vec{\nabla}\times(\vec{A}+\vec{B}) = \vec{\nabla}\times\vec{A}+\vec{\nabla}\times\vec{B}$
$\vec{\nabla}\cdot(\psi\vec{A}) = \vec{A}\cdot\vec{\nabla}\psi+\psi\vec{\nabla}\cdot\vec{A}$
$\vec{\nabla}\times(\psi\vec{A}) = \vec{\nabla}\psi\times\vec{A}+\psi\,\vec{\nabla}\times\vec{A}$
$\vec{\nabla}(\vec{A}\cdot\vec{B}) = (\vec{A}\cdot\vec{\nabla})\vec{B}+(\vec{B}\cdot\vec{\nabla})\vec{A}+\vec{A}\times(\vec{\nabla}\times\vec{B})+\vec{B}\times(\vec{\nabla}\times\vec{A})$
$\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot\vec{\nabla}\times\vec{A}-\vec{A}\cdot\vec{\nabla}\times\vec{B}$
$\vec{\nabla}\times(\vec{A}\times\vec{B}) = \vec{A}\vec{\nabla}\cdot\vec{B}-\vec{B}\vec{\nabla}\cdot\vec{A}+(\vec{B}\cdot\vec{\nabla})\vec{A}-(\vec{A}\cdot\vec{\nabla})\vec{B}$
$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A})-\vec{\nabla}^2\vec{A}$

Produit scalaire des vecteurs unitaires (rectangulaires/cylindrique)

	$\hat{a}_\rho$	$\hat{a}_\phi$	$\hat{a}_z$
$\hat{a}_x$	$\cos\phi$	$-\sin\phi$	0
$\hat{a}_y$	$\sin\phi$	$\cos\phi$	0
$\hat{a}_z$	0	0	1

Produit scalaire des vecteurs unitaires (rectangulaires/sphérique)

	$\hat{a}_r$	$\hat{a}_\theta$	$\hat{a}_\phi$
$\hat{a}_x$	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$	$-\sin\phi$
$\hat{a}_y$	$\sin\theta\sin\phi$	$\cos\theta\sin\phi$	$\cos\phi$
$\hat{a}_z$	$\cos\theta$	$-\sin\theta$	0