# Module : 4 Stress-Strain Relations

## 4.1.1 INTRODUCTION

In the previous chapters, the state of stress at a point was defined in terms of six components of stress, and in addition three equilibrium equations were developed to relate the internal stresses and the applied forces. These relationships were independent of the deformations (strains) and the material behaviour. Hence, these equations are applicable to all types of materials.

Also, the state of strain at a point was defined in terms of six components of strain. These six strain-displacement relations and compatibility equations were derived in order to relate uniquely the strains and the displacements at a point. These equations were also independent of the stresses and the material behavior and hence are applicable to all materials.

Irrespective of the independent nature of the equilibrium equations and strain-displacement relations, in practice, it is essential to study the general behaviour of materials under applied loads including these relations. This becomes necessary due to the application of a load, stresses, deformations and hence strains will develop in a body. Therefore in a general three-dimensional system, there will be 15 unknowns namely 3 displacements, 6 strains and 6 stresses. In order to determine these 15 unknowns, we have only 9 equations such as 3 equilibrium equations and 6 strain-displacement equations. It is important to note that the compatibility conditions as such cannot be used to determine either the displacements or strains. Hence the additional six equations have to be based on the relationships between six stresses and six strains. These equations are known as "Constitutive equations" because they describe the macroscopic behavior of a material based on its internal constitution.

## 4.1.2 LINEAR ELASTICITY-GENERALISED HOOKE'S LAW

There is a unique relationship between stress and strain defined by Hooke's Law, which is independent of time and loading history. The law assumes that all the strain changes resulting from stress changes are instantaneous and the system is completely reversible and all the input energy is recovered in unloading.

In case of uniaxial loading, stress is related to strain as

$$\sigma_x = E\varepsilon_x \tag{4.0}$$

where $E$ is known as "Modulus of Elasticity".

The above expression is applicable within the linear elastic range and is called Hooke's Law.

In general, each strain is dependent on each stress. For example, the strain $\varepsilon_x$ written as a function of each stress is

$$\varepsilon_x = C_{11}\sigma_x + C_{12}\sigma_y + C_{13}\sigma_z + C_{14}\tau_{xy} + C_{15}\tau_{yz} + C_{16}\tau_{zx} + C_{17}\tau_{xz} + C_{18}\tau_{zy} + C_{19}\tau_{yx} \tag{4.1}$$

Similarly, stresses can be expressed in terms of strains stating that at each point in a material, each stress component is linearly related to all the strain components. This is known as "Generalised Hook's Law".

For the most general case of three-dimensional state of stress, equation (4.0) can be written as

$$\{\sigma_{ij}\} = (D_{ijkl})\,\{\varepsilon_{kl}\} \tag{4.2}$$

where $(D_{ijkl})$ = Elasticity matrix

$\{\sigma_{ij}\}$ = Stress components

$\{\varepsilon_{kl}\}$ = Strain components

Since both stress $\sigma_{ij}$ and strain $\varepsilon_{ij}$ are second-order tensors, it follows that $D_{ijkl}$ is a fourth order tensor, which consists of $3^4$ = 81 material constants if symmetry is not assumed. Therefore in matrix notation, the stress-strain relations would be

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xz} \\ \tau_{zy} \\ \tau_{yx} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16} & D_{17} & D_{18} & D_{19} \\ D_{21} & D_{22} & D_{23} & D_{24} & D_{25} & D_{26} & D_{27} & D_{28} & D_{29} \\ D_{31} & D_{32} & D_{33} & D_{34} & D_{35} & D_{36} & D_{37} & D_{38} & D_{39} \\ D_{41} & D_{42} & D_{43} & D_{44} & D_{45} & D_{46} & D_{47} & D_{48} & D_{49} \\ D_{51} & D_{52} & D_{53} & D_{54} & D_{55} & D_{56} & D_{57} & D_{58} & D_{59} \\ D_{61} & D_{62} & D_{63} & D_{64} & D_{65} & D_{66} & D_{67} & D_{68} & D_{69} \\ D_{71} & D_{72} & D_{73} & D_{74} & D_{75} & D_{76} & D_{77} & D_{78} & D_{79} \\ D_{81} & D_{82} & D_{83} & D_{84} & D_{85} & D_{86} & D_{87} & D_{88} & D_{89} \\ D_{91} & D_{92} & D_{93} & D_{94} & D_{95} & D_{96} & D_{97} & D_{98} & D_{99} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xz} \\ \gamma_{zy} \\ \gamma_{yx} \end{Bmatrix} \tag{4.3}$$

Now, from $\sigma_{ij} = \sigma_{ji}$ and $\varepsilon_{ij} = \varepsilon_{ji}$ the number of 81 material constants is reduced to 36 under symmetric conditions of $D_{ijkl} = D_{jikl} = D_{ijlk} = D_{jilk}$

Therefore in matrix notation, the stress – strain relations can be

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16} \\ D_{21} & D_{22} & D_{23} & D_{24} & D_{25} & D_{26} \\ D_{31} & D_{32} & D_{33} & D_{34} & D_{35} & D_{36} \\ D_{41} & D_{42} & D_{43} & D_{44} & D_{45} & D_{46} \\ D_{51} & D_{52} & D_{53} & D_{54} & D_{55} & D_{56} \\ D_{61} & D_{62} & D_{63} & D_{64} & D_{65} & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} \tag{4.4}$$

Equation (4.4) indicates that 36 elastic constants are necessary for the most general form of anisotropy (different elastic properties in all directions). It is generally accepted, however, that the stiffness matrix $D_{ij}$ is symmetric, in which case the number of independent elastic constants will be reduced to 21. This can be shown by assuming the existence of a strain energy function ***U***.

It is often desired in classical elasticity to have a potential function

$$U = U(\varepsilon_{ij}) \tag{4.5}$$

with the property that

$$\frac{\partial U}{\partial \varepsilon_{ij}} = \sigma_{ij} \tag{4.6}$$

Such a function is called a "strain energy" or "strain energy density function".

By equation (4.6), we can write

$$\frac{\partial U}{\partial \varepsilon_i} = \sigma_i = D_{ij}\varepsilon_j \tag{4.7}$$

Differentiating equation (4.7) with respect to $\varepsilon_j$ , then

$$\frac{\partial^2 U}{\partial \varepsilon_i \partial \varepsilon_j} = D_{ij} \tag{4.8}$$

The free index in equation (4.7) can be changed so that

$$\frac{\partial U}{\partial \varepsilon_j} = \sigma_j = D_{ji}\varepsilon_i \tag{4.9}$$

Differentiating equation (4.9) with respect to $\varepsilon_i$ , then,

$$\frac{\partial^2 U}{\partial \varepsilon_j \varepsilon_i} = D_{ji} \tag{4.10}$$

Hence, equations (4.8) and (4.10) are equal, or $D_{ij} = D_{ji}$

which implies that $D_{ij}$ is symmetric. Then most general form of the stiffness matrix or array becomes

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16} \\ D_{12} & D_{22} & D_{23} & D_{24} & D_{25} & D_{26} \\ D_{13} & D_{23} & D_{33} & D_{34} & D_{35} & D_{36} \\ D_{14} & D_{24} & D_{34} & D_{44} & D_{45} & D_{46} \\ D_{15} & D_{25} & D_{35} & D_{45} & D_{55} & D_{56} \\ D_{16} & D_{26} & D_{36} & D_{46} & D_{56} & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} \tag{4.11}$$

Or

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16} \\ & D_{22} & D_{23} & D_{24} & D_{25} & D_{26} \\ & & D_{33} & D_{34} & D_{35} & D_{36} \\ & & & D_{44} & D_{45} & D_{46} \\ & & & & D_{55} & D_{56} \\ & & & & & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} \tag{4.12}$$

Further, a material that exhibits symmetry with respect to three mutually orthogonal planes is called an "orthotropic" material. If the *xy*, *yz* and *zx* planes are considered planes of symmetry, then equation (4.11) reduces to 12 elastic constants as below.

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & 0 & 0 & 0 \\ D_{21} & D_{22} & D_{23} & 0 & 0 & 0 \\ D_{31} & D_{32} & D_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & D_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} \tag{4.13}$$

Also, due to orthotropic symmetry, the number of material constants for a linear elastic orthotropic material reduces to 9 as shown below.

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{xy}\\ \tau_{yz}\\ \tau_{zx}\end{Bmatrix}=\begin{bmatrix}D_{11} & D_{12} & D_{13} & 0 & 0 & 0\\ & D_{22} & D_{23} & 0 & 0 & 0\\ & & D_{33} & 0 & 0 & 0\\ & & & D_{44} & 0 & 0\\ & & & & D_{55} & 0\\ & & & & & D_{66}\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{xy}\\ \gamma_{yz}\\ \gamma_{zx}\end{Bmatrix} \tag{4.14}$$

Now, in the case of a transversely isotropic material, the material exhibits a rationally elastic symmetry about one of the coordinate axes, $x$, $y$, and $z$. In such case, the material constants reduce to 8 as shown below.

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{xy}\\ \tau_{yz}\\ \tau_{zx}\end{Bmatrix}=\begin{bmatrix}D_{11} & D_{12} & D_{13} & 0 & 0 & 0\\ & D_{22} & D_{23} & 0 & 0 & 0\\ & & D_{33} & 0 & 0 & 0\\ & & & \frac{1}{2}(D_{11}-D_{12}) & 0 & 0\\ Symmetry & & & & D_{55} & 0\\ & & & & & D_{66}\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{xy}\\ \gamma_{yz}\\ \gamma_{zx}\end{Bmatrix} \tag{4.15}$$

Further, for a linearly elastic material with cubic symmetry for which the properties along the $x$, $y$ and $z$ directions are identical, there are only 3 independent material constants. Therefore, the matrix form of the stress – strain relation can be expressed as:

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{xy}\\ \tau_{yz}\\ \tau_{zx}\end{Bmatrix}=\begin{bmatrix}D_{11} & D_{12} & D_{12} & 0 & 0 & 0\\ & D_{11} & D_{12} & 0 & 0 & 0\\ & & D_{11} & 0 & 0 & 0\\ & & & D_{44} & 0 & 0\\ Symmetry & & & & D_{44} & 0\\ & & & & & D_{44}\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{xy}\\ \gamma_{yz}\\ \gamma_{zx}\end{Bmatrix} \tag{4.16}$$

## 4.1.3 ISOTROPY

For a material whose elastic properties are not a function of direction at all, only two independent elastic material constants are sufficient to describe it’s behavior completely. This material is called "Isotropic linear elastic". The stress- strain relationship for this material is thus written as an extension of that of a transversely isotropic material as shown below.

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{12} & 0 & 0 & 0 \\ & D_{11} & D_{12} & 0 & 0 & 0 \\ & & D_{11} & 0 & 0 & 0 \\ & & & \frac{1}{2}(D_{11}-D_{12}) & 0 & 0 \\ Symmetry & & & & \frac{1}{2}(D_{11}-D_{12}) & 0 \\ & & & & & \frac{1}{2}(D_{11}-D_{12}) \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} \tag{4.17}$$

Thus, we get only 2 independent elastic constants.

Replacing $D_{12}$ and $\frac{1}{2}(D_{11} - D_{12})$ respectively by $\lambda$ and $G$ which are called "Lame's constants", where $G$ is also called shear modulus of elasticity, equation (4.17) can be written as:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \begin{bmatrix} 2G+\lambda & \lambda & \lambda & 0 & 0 & 0 \\ & 2G+\lambda & \lambda & 0 & 0 & 0 \\ & & 2G+\lambda & 0 & 0 & 0 \\ & & & G & 0 & 0 \\ Symmetry & & & & G & 0 \\ & & & & & G \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} \tag{4.18}$$

Therefore, the stress-strain relationships may be expressed as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \begin{bmatrix} 2G+\lambda & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & 2G+\lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & 2G+\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} \tag{4.19}$$

Therefore,

$$\begin{aligned} \sigma_x &= (2G+\lambda)\,\varepsilon_x + \lambda(\varepsilon_y + \varepsilon_z) \\ \sigma_y &= (2G+\lambda)\varepsilon_y + \lambda(\varepsilon_z + \varepsilon_x) \\ \sigma_z &= (2G+\lambda)\varepsilon_z + \lambda(\varepsilon_x + \varepsilon_y) \end{aligned} \tag{4.20}$$

Also, $\tau_{xy} = G\gamma_{xy}$

$\tau_{yz} = G\gamma_{yz}$

$\tau_{zx} = G\gamma_{zx}$

Now, expressing strains in terms of stresses, we get

$$\varepsilon_x = \frac{\lambda + G}{G(3\lambda + 2G)}\sigma_x - \frac{\lambda}{2G(3\lambda + 2G)}(\sigma_y + \sigma_z)$$

$$\varepsilon_y = \frac{\lambda + G}{G(3\lambda + 2G)}\sigma_y - \frac{\lambda}{2G(3\lambda + 2G)}(\sigma_z + \sigma_x)$$

$$\varepsilon_z = \frac{\lambda + G}{G(3\lambda + 2G)}\sigma_z - \frac{\lambda}{2G(3\lambda + 2G)}(\sigma_x + \sigma_y) \qquad (4.21)$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$

$$\gamma_{yz} = \frac{\tau_{yz}}{G}$$

$$\gamma_{zx} = \frac{\tau_{zx}}{G}$$

Now consider a simple tensile test

Therefore,

$$\varepsilon_x = \frac{\sigma_x}{E} = \frac{\lambda + G}{G(3\lambda + 2G)}\sigma_x$$

or $$\frac{1}{E} = \frac{\lambda + G}{G(3\lambda + 2G)}$$

or $$E = \frac{G(3\lambda + 2G)}{(\lambda + G)} \qquad (4.22)$$

where $E$ = Modulus of Elasticity

Also,

$$\varepsilon_y = -\nu\varepsilon_x = -\nu\frac{\sigma_x}{E}$$

$$\varepsilon_z = -\nu\varepsilon_x = -\nu\frac{\sigma_x}{E}$$

where $\nu$ = Poisson's ratio

For $\sigma_y = \sigma_z = 0$, we get from equation (4.21)

$$-\frac{\lambda}{2G(3\lambda+2G)}\ \sigma_x = -\frac{\nu}{E}\ \sigma_x$$

Therefore, $\dfrac{\nu}{E} = \dfrac{\lambda}{2G(3\lambda+2G)}$ (4.23)

Substituting the value of $E$ from equation (4.22), we get

$$\frac{\nu(\lambda+G)}{G(3\lambda+2G)} = \frac{\lambda}{2G(3\lambda+2G)}$$

Therefore, $2\nu(\lambda+G) = \lambda$

or $\nu = \dfrac{\lambda}{2(\lambda+G)}$ (4.24)

Solving for λ from equations (4.22) and (4.23), we get

$$\lambda = \frac{G(2G-E)}{(E-3G)} = \frac{4G^2\nu}{(E-6G\nu)}$$

or $G = \dfrac{E}{2(1+\nu)}$ (4.25)

For a hydrostatic state of stress, i.e., all round compression $p$,

$\sigma_x = \sigma_y = \sigma_z = -p$

Therefore, $\varepsilon_x+\varepsilon_y+\varepsilon_z = \dfrac{-3(1-2\nu)p}{E}$

or $-p = \dfrac{E(\varepsilon_x+\varepsilon_y+\varepsilon_z)}{3(1-2\nu)}$

$= (\lambda + \dfrac{2G}{3})(\varepsilon_x+\varepsilon_y+\varepsilon_z)$

or $-p = K(\varepsilon_x+\varepsilon_y+\varepsilon_z)$

Hence, $K = \left(\lambda+\dfrac{2G}{3}\right)$ (4.26)

where $K$ = Bulk modulus of elasticity.

Also,

$-p = K(\varepsilon_x+\varepsilon_y+\varepsilon_z)$

$$-p = K\left[\frac{-3p(1-2\nu)}{E}\right]$$

or $$E = K\left[3(1-2\nu)\right]$$

Therefore, $$K = \frac{E}{3(1-2\nu)} \tag{4.27}$$