Module 5: Two Dimensional Problems in Cartesian Coordinate System

5.2.1 THE STRESS FUNCTION

For two-dimensional problems without considering the body forces, the equilibrium equations are given by

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = 0$$

and the equation of compatibility is

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = 0$$

The equations of equilibrium are identically satisfied by the stress function, $\phi(x, y)$, introduced by G. B. Airy for the two dimensional case. The relationships between the stress function ϕ and the stresses are as follows:

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 \phi}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} \tag{5.11}$$

Substituting the above expressions into the compatibility equation, we get

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0 \tag{5.12}$$

Further, equilibrium equations are automatically satisfied by substituting the above expressions for stress components.

In general, $\nabla^4 \phi = 0$

where $\nabla^4 = \frac{\partial^4}{\partial x^4} + \frac{2\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$

The above Equation (5.12) is known as "Biharmonic equation" for plane stress and plane strain problems. Since the Biharmonic equation satisfies all the equilibrium and compatibility equations, a solution to this equation is also the solution for a two-dimensional problem. However, the solution in addition to satisfying the Biharmonic equation also has to satisfy the boundary conditions.

To solve the derived equations of elasticity, it is suggested to use polynomial functions, inverse functions or semi-inverse functions. The use of polynomial functions for solving two-dimensional problems is discussed in the next article. The inverse method requires examination of the assumed solutions with a view towards finding one which will satisfy the governing equations and the boundary conditions.

The semi-inverse method requires the assumption of a partial solution, formed by expressing stress, strain, displacement, or stress function in terms of known or undetermined coefficients. The governing equations are thus rendered more manageable.