Module 6: Two Dimensional Problems in Polar Coordinate System

6.2.1 AXISYMMETRIC PROBLEMS

Many engineering problems involve solids of revolution subjected to axially symmetric loading. The examples are a circular cylinder loaded by uniform internal or external pressure or other axially symmetric loading (Figure 6.4a), and a semi-infinite half space loaded by a circular area, for example a circular footing on a soil mass (Figure 6.4b). It is convenient to express these problems in terms of the cylindrical co-ordinates. Because of symmetry, the stress components are independent of the angular (θ) co-ordinate; hence, all derivatives with respect to θ vanish and the components v, $\gamma_{r\theta}$, $\gamma_{\theta z}$, $\tau_{r\theta}$ and $\tau_{\theta z}$ are zero. The non-zero stress components are σ_r, σ_θ, σ_z and τ_{rz}.

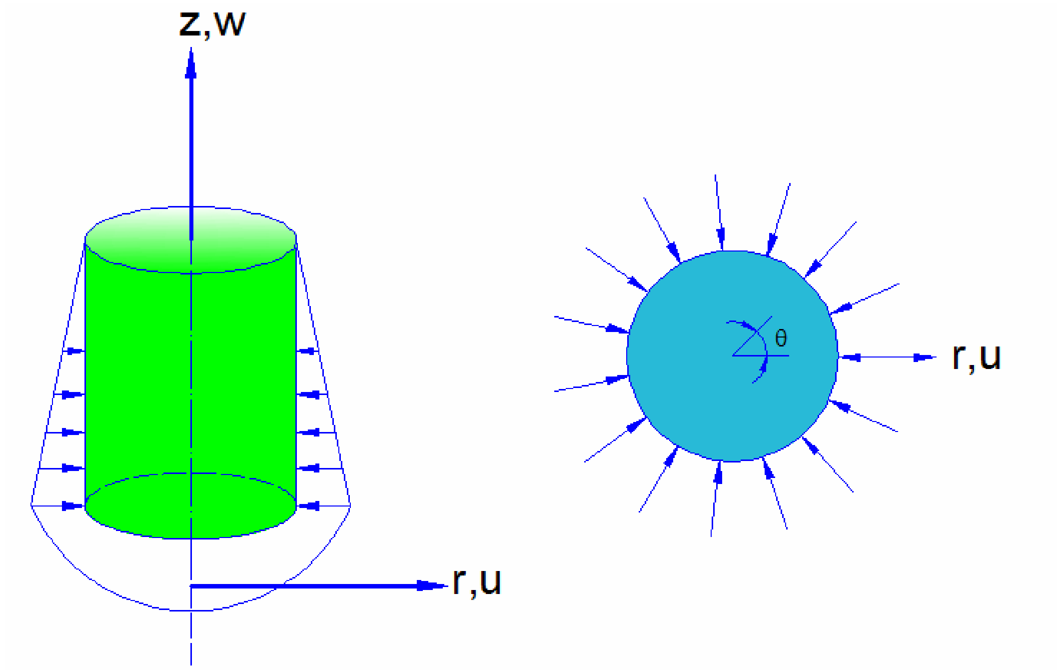
The strain-displacement relations for the non-zero strains become

$$\varepsilon_r = \frac{\partial u}{\partial r}, \varepsilon_\theta = \frac{u}{r}, \varepsilon_z = \frac{\partial w}{\partial z}$$

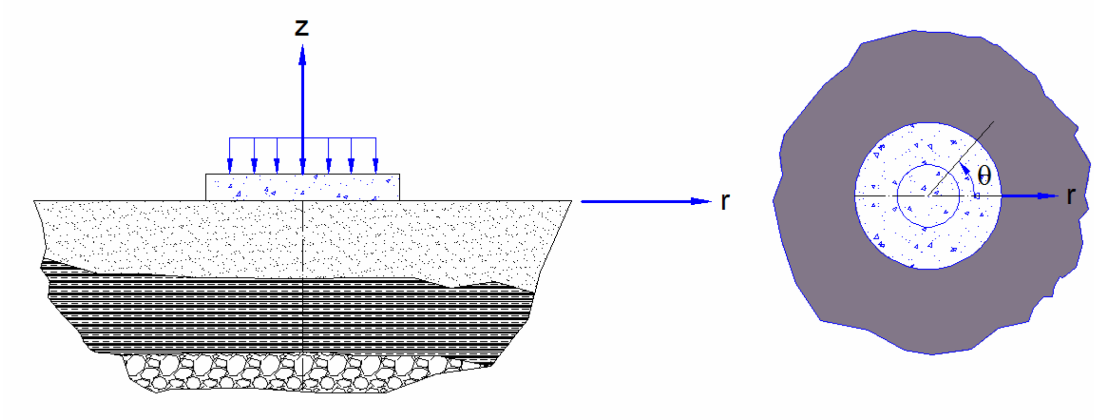
$$\gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \tag{6.19}$$

and the constitutive relation is given by

$$\begin{Bmatrix} \sigma_r \\ \sigma_z \\ \sigma_\theta \\ \tau_{rz} \end{Bmatrix} = \frac{E}{(1+v)(1-2v)} \begin{bmatrix} (1-v) & v & v & 0 \\ & (1-v) & v & 0 \\ & & (1-v) & 0 \\ & \text{Symmetry} & & \frac{(1-2v)}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{Bmatrix}$$



(a) Cylinder under axisymmetric loading



(b) Circular Footing on Soil mass

Figure 6.4 Axisymmetric Problems

6.2.2 Thick-Walled Cylinder Subjected to Internal and External Pressures

Consider a cylinder of inner radius 'a' and outer radius 'b' as shown in the figure 6.5. Let the cylinder be subjected to internal pressure p_i and an external pressure p_0. This problem can be treated either as a plane stress case (σ_z = 0) or as a plane strain case (ε_z = 0).

Case (a): Plane Stress

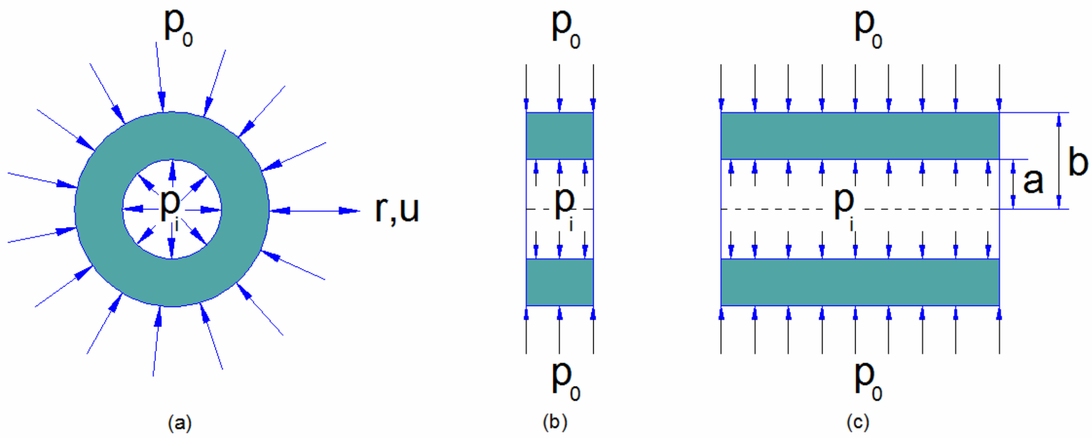


Figure 6.5 (a) Thick-walled cylinder (b) Plane stress case (c) Plane strain case

Consider the ends of the cylinder which are free to expand. Let σ_z = 0. Owing to uniform radial deformation, τ_{rz} = 0. Neglecting the body forces, equation of equilibrium reduces to

$$\frac{\partial \sigma_r}{\partial r} + \left(\frac{\sigma_r - \sigma_\theta}{r} \right) = 0 \tag{6.20}$$

Here σ_θ and σ_r denote the tangential and radial stresses acting normal to the sides of the element.

Since r is the only independent variable, the above equation can be written as

$$\frac{d}{dr}(r\sigma_r) - \sigma_\theta = 0 \tag{6.21}$$

From Hooke's Law,

$$\varepsilon_r = \frac{1}{E}(\sigma_r - \nu\sigma_\theta), \quad \varepsilon_\theta = \frac{1}{E}(\sigma_\theta - \nu\sigma_r)$$

Therefore, $\varepsilon_r = \dfrac{du}{dr}$ and $\varepsilon_\theta = \dfrac{u}{r}$ or the stresses in terms of strains are

$$\sigma_r = \frac{E}{(1-\nu^2)}(\varepsilon_r + \nu\varepsilon_\theta)$$

$$\sigma_\theta = \frac{E}{(1-\nu^2)}(\varepsilon_\theta + \nu\varepsilon_r)$$

Substituting the values of ε_r and ε_θ in the above expressions, we get

$$\sigma_r = \frac{E}{(1-\nu^2)}\left(\frac{du}{dr} + \nu\frac{u}{r}\right)$$

$$\sigma_\theta = \frac{E}{(1-\nu^2)}\left(\frac{u}{r} + \nu\frac{du}{dr}\right)$$

Substituting these in the equilibrium Equation (6.21)

$$\frac{d}{dr}\left(r\frac{du}{dr} + \nu u\right) - \left(\frac{u}{r} + \nu\frac{du}{dr}\right) = 0$$

$$\frac{du}{dr} + r\frac{d^2u}{dr^2} + \nu\frac{du}{dr} - \frac{u}{r} - \nu\frac{du}{dr} = 0$$

$$\text{or} \quad \frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0$$

The above equation is called equidimensional equation in radial displacement. The solution of the above equation is

$$u = C_1 r + C_2/r \tag{6.22}$$

where C_1 and C_2 are constants.

The radial and tangential stresses are written in terms of constants of integration C_1 and C_2.

Therefore,

$$\sigma_r = \frac{E}{(1-\nu^2)}\left[C_1(1+\nu) - C_2\left(\frac{1-\nu}{r^2}\right)\right]$$

$$\sigma_\theta = \frac{E}{(1-\nu^2)}\left[C_1(1+\nu) + C_2\left(\frac{1-\nu}{r^2}\right)\right] \tag{6.23}$$

The constants are determined from the boundary conditions.

$$\begin{aligned} \text{when} \quad & r = a, \quad \sigma_r = -p_i \\ & r = b, \quad \sigma_r = -p_0 \end{aligned} \tag{6.23a}$$

Hence, $\frac{E}{(1-\nu^2)}\left[C_1(1+\nu)-C_2\left(\frac{1-\nu}{a^2}\right)\right] = -p_i$

and $\frac{E}{(1-\nu^2)}\left[C_1(1+\nu)-C_2\left(\frac{1-\nu}{b^2}\right)\right] = -p_0$

where the negative sign in the boundary conditions denotes compressive stress.

The constants are evaluated by substitution of equation (6.23a) into (6.23)

$$C_1 = \left(\frac{1-\nu}{E}\right)\left(\frac{a^2 p_i - b^2 p_0}{(b^2-a^2)}\right)$$

$$C_2 = \left(\frac{1+\nu}{E}\right)\left(\frac{a^2 b^2 (p_i - p_0)}{(b^2-a^2)}\right)$$

Substituting these in Equations (6.22) and (6.23), we get

$$\sigma_r = \left(\frac{a^2 p_i - b^2 p_0}{b^2-a^2}\right) - \left(\frac{(p_i - p_0)a^2 b^2}{(b^2-a^2)r^2}\right) \tag{6.24}$$

$$\sigma_\theta = \left(\frac{a^2 p_i - b^2 p_0}{b^2-a^2}\right) + \left(\frac{(p_i - p_0)a^2 b^2}{(b^2-a^2)r^2}\right) \tag{6.25}$$

$$u = \left(\frac{1-\nu}{E}\right)\frac{(a^2 p_i - b^2 p_0)r}{(b^2-a^2)} + \left(\frac{1+\nu}{E}\right)\frac{(p_i - p_0)a^2 b^2}{(b^2-a^2)r} \tag{6.26}$$

These expressions were first derived by G. Lambe.

It is interesting to observe that the sum ($\sigma_r + \sigma_\theta$) is constant through the thickness of the wall of the cylinder, regardless of radial position. Hence according to Hooke's law, the stresses σ_r and σ_θ produce a uniform extension or contraction in z-direction. The cross-sections perpendicular to the axis of the cylinder remain plane. If two adjacent cross-sections are considered, then the deformation undergone by the element does not interfere with the deformation of the neighbouring element. Hence, the elements are considered to be in the plane stress state.

Special Cases

(i) ***A cylinder subjected to internal pressure only:*** In this case, $p_0 = 0$ and $p_i = p$.

Then Equations (6.24) and (6.25) become

$$\sigma_r = \frac{pa^2}{(b^2-a^2)}\left(1-\frac{b^2}{r^2}\right) \tag{6.27}$$

$$\sigma_\theta = \frac{pa^2}{(b^2-a^2)}\left(1+\frac{b^2}{r^2}\right) \tag{6.28}$$

Figure 6.6 shows the variation of radial and circumferential stresses across the thickness of the cylinder under internal pressure.

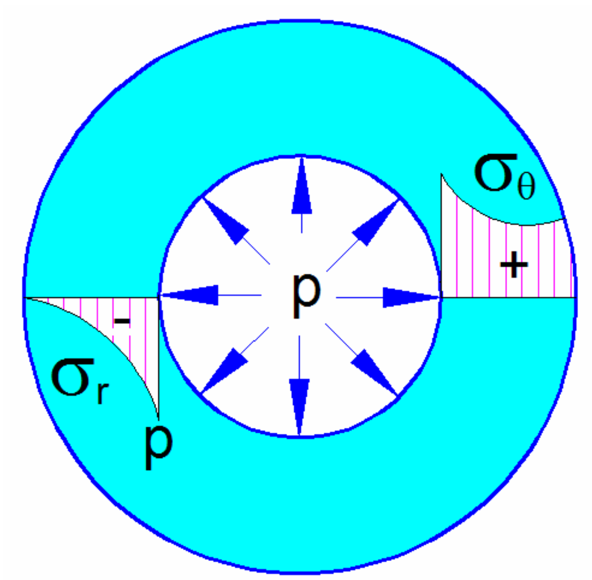


Figure 6.6 Cylinder subjected to internal pressure

The circumferential stress is greatest at the inner surface of the cylinder and is given by

$$(\sigma_\theta)_{max} = \frac{p(a^2+b^2)}{b^2-a^2} \tag{6.29}$$

(ii) ***A cylinder subjected to external pressure only:*** In this case, $p_i = 0$ and $p_0 = p$.

Equation (6.25) becomes

$$\sigma_r = -\left(\frac{pb^2}{b^2-a^2}\right)\left(1-\frac{a^2}{r^2}\right) \tag{6.30}$$

$$\sigma_\theta = -\left(\frac{pb^2}{b^2-a^2}\right)\left(1+\frac{a^2}{r^2}\right) \tag{6.31}$$

Figure 6.7 represents the variation of σ_r and σ_θ across the thickness.

However, if there is no inner hole, i.e., if $a = 0$, the stresses are uniformly distributed in the cylinder as

$\sigma_r = \sigma_\theta = -p$

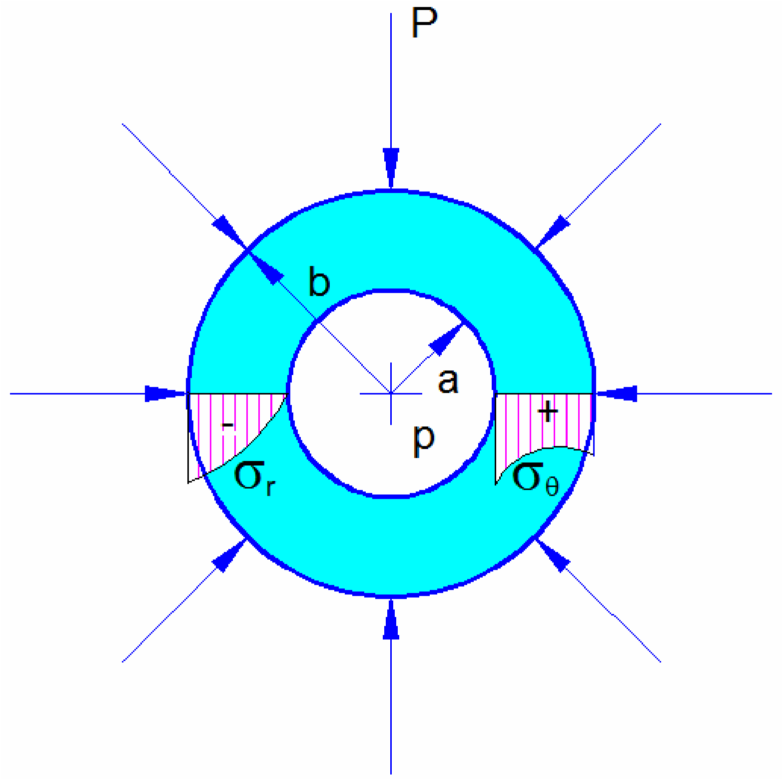


Figure 6.7 Cylinder subjected to external pressure

Case (b): Plane Strain

If a long cylinder is considered, sections that are far from the ends are in a state of plane strain and hence σ_z does not vary along the z-axis.

Now, from Hooke's Law,

$$\varepsilon_r = \frac{1}{E}\left[\sigma_r - \nu\left(\sigma_\theta + \sigma_z\right)\right]$$

$$\varepsilon_\theta = \frac{1}{E}\left[\sigma_\theta - \nu\left(\sigma_r + \sigma_z\right)\right]$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu\left(\sigma_r + \sigma_\theta\right)\right]$$

Since $\varepsilon_z = 0$, then

$$0 = \frac{1}{E}\left[\sigma_z - \nu\left(\sigma_r + \sigma_\theta\right)\right]$$

$\sigma_z = \nu\,(\sigma_r + \sigma_\theta)$

Hence,

$$\varepsilon_r = \frac{(1+\nu)}{E}\left[(1-\nu)\sigma_r - \nu\sigma_\theta\right]$$

$$\varepsilon_\theta = \frac{(1+\nu)}{E}\left[(1-\nu)\sigma_\theta - \nu\sigma_r\right]$$

Solving for σ_θ and σ_r,

$$\sigma_\theta = \frac{E}{(1-2\nu)(1+\nu)}\left[\nu\varepsilon_r + (1-\nu)\varepsilon_\theta\right]$$

$$\sigma_r = \frac{E}{(1-2\nu)(1+\nu)}\left[(1-\nu)\varepsilon_r + \nu\varepsilon_\theta\right]$$

Substituting the values of ε_r and ε_θ, the above expressions for σ_θ and σ_r can be written as

$$\sigma_\theta = \frac{E}{(1-2\nu)(1+\nu)}\left[\nu\frac{du}{dr} + (1-\nu)\frac{u}{r}\right]$$

$$\sigma_r = \frac{E}{(1-2\nu)(1+\nu)}\left[(1-\nu)\frac{du}{dr} + \frac{\nu u}{r}\right]$$

Substituting these in the equation of equilibrium (6.21), we get

$$\frac{d}{dr}\left[(1-\nu)r\frac{du}{dr} + \nu u\right] - \nu\frac{du}{dr} - (1-\nu)\frac{u}{r} = 0$$

or $\dfrac{du}{dr} + r\dfrac{d^2u}{dr^2} - \dfrac{u}{r} = 0$

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0$$

The solution of this equation is the same as in Equation (6.22)

$u = C_1 r + C_2/r$

where C_1 and C_2 are constants of integration. Therefore, σ_θ and σ_r are given by

$$\sigma_\theta = \frac{E}{(1-2\nu)(1+\nu)}\left[C_1 + (1-2\nu)\frac{C_2}{r^2}\right]$$

$$\sigma_r = \frac{E}{(1-2\nu)(1+\nu)}\left[C_1 - (1-2\nu)\frac{C_2}{r^2}\right]$$

Applying the boundary conditions,

$\sigma_r = -\,p_i$ when $r = a$

$\sigma_r = -\,p_0$ when $r = b$

Therefore, $\dfrac{E}{(1-2\nu)(1+\nu)}\left[C_1 - (1-2\nu)\dfrac{C_2}{a^2}\right] = -p_i$

$$\frac{E}{(1-2\nu)(1+\nu)}\left[C_1 - (1-2\nu)\frac{C_2}{b^2}\right] = -p_o$$

Solving, we get

$$C_1 = \frac{(1-2\nu)(1+\nu)}{E}\left(\frac{p_0 b^2 - p_i a^2}{a^2 - b^2}\right)$$

and $\quad C_2 = \dfrac{(1+\nu)}{E}\left(\dfrac{(p_0 - p_i)a^2 b^2}{a^2 - b^2}\right)$

Substituting these, the stress components become

$$\sigma_r = \left(\frac{p_i a^2 - p_0 b^2}{b^2 - a^2}\right) - \left(\frac{p_i - p_0}{b^2 - a^2}\right)\frac{a^2 b^2}{r^2} \tag{6.32}$$

$$\sigma_\theta = \left(\frac{p_i a^2 - p_0 b^2}{b^2 - a^2}\right) + \left(\frac{p_i - p_0}{b^2 - a^2}\right)\frac{a^2 b^2}{r^2} \tag{6.33}$$

$$\sigma_z = 2\nu\left(\frac{p_0 a^2 - p_i b^2}{b^2 - a^2}\right) \tag{6.34}$$

It is observed that the values of σ_r and σ_θ are identical to those in plane stress case. But in plane stress case, $\sigma_z = 0$, whereas in the plane strain case, σ_z has a constant value given by equation (6.34).

6.2.3 ROTATING DISKS OF UNIFORM THICKNESS

The equation of equilibrium given by

$$\frac{d\sigma_r}{dr}+\left(\frac{\sigma_r-\sigma_\theta}{r}\right)+F_r=0 \tag{a}$$

is used to treat the case of a rotating disk, provided that the centrifugal "inertia force" is included as a body force. It is assumed that the stresses induced by rotation are distributed symmetrically about the axis of rotation and also independent of disk thickness. Thus, application of equation (a), with the body force per unit volume F_r equated to the centrifugal force $\rho w^2 r$, yields

$$\frac{d\sigma_r}{dr}+\left(\frac{\sigma_r-\sigma_\theta}{r}\right)+\rho w^2 r=0 \tag{6.35}$$

where ρ is the mass density and w is the constant angular speed of the disk in rad/sec. The above equation (6.35) can be written as

$$\frac{d}{dr}(r\sigma_r)-\sigma_\theta+\rho w^2 r^2=0 \tag{6.36}$$

But the strain components are given by

$$\varepsilon_r=\frac{du}{dr} \quad \text{and} \quad \varepsilon_\theta=\frac{u}{r} \tag{6.37}$$

From Hooke's Law, with $\sigma_z = 0$

$$\varepsilon_r=\frac{1}{E}(\sigma_r-\nu\sigma_\theta) \tag{6.38}$$

$$\varepsilon_\theta=\frac{1}{E}(\sigma_\theta-\nu\sigma_r) \tag{6.39}$$

From equation (6.37),

$$u=r\varepsilon_\theta$$

$$\frac{du}{dr}=\varepsilon_r=\frac{d}{dr}(r\varepsilon_\theta)$$

Using Hooke's Law, we can write equation (6.38) as

$$\frac{1}{E}(\sigma_r-\nu\sigma_\theta)=\frac{1}{E}\left[\frac{d}{dr}(r\sigma_\theta-\nu r\sigma_r)\right] \tag{6.40}$$

Let $r\,\sigma_r = y$ (6.41)

Then from equation (6.36)

$$\sigma_\theta=\frac{dy}{dr}+\rho w^2 r^2 \tag{6.42}$$

Substituting these in equation (6.40), we obtain

$$r^2 \frac{d^2 y}{dr^2} + r\frac{dy}{dr} - y + (3+\nu) + \rho w^2 r^3 = 0$$

The solution of the above differential equation is

$$y = Cr + C_1 \left(\frac{1}{r}\right) - \left(\frac{3+\nu}{8}\right)\rho w^2 r^3 \tag{6.43}$$

From Equations (6.41) and (6.42), we obtain

$$\sigma_r = C + C_1 \left(\frac{1}{r^2}\right) - \left(\frac{3+\nu}{8}\right)\rho w^2 r^2 \tag{6.44}$$

$$\sigma_\theta = C - C_1 \left(\frac{1}{r^2}\right) - \left(\frac{1+3\nu}{8}\right)\rho w^2 r^2 \tag{6.45}$$

The constants of integration are determined from the boundary conditions.

6.2.4 SOLID DISK

For a solid disk, it is required to take $C_1 = 0$, otherwise, the stresses σ_r and σ_θ becomes infinite at the centre. The constant C is determined from the condition at the periphery ($r = b$) of the disk. If there are no forces applied, then

$$(\sigma_r)_{r=b} = C - \left(\frac{3+\nu}{8}\right)\rho w^2 b^2 = 0$$

Therefore, $C = \left(\frac{3+\nu}{8}\right)\rho w^2 b^2$ (6.46)

Hence, Equations (6.44) and (6.45) become,

$$\sigma_r = \left(\frac{3+\nu}{8}\right)\rho w^2 (b^2 - r^2) \tag{6.47}$$

$$\sigma_\theta = \left(\frac{3+\nu}{8}\right)\rho w^2 b^2 - \left(\frac{1+3\nu}{8}\right)\rho w^2 r^2 \tag{6.48}$$

The stresses attain their maximum values at the centre of the disk, i.e., at $r = 0$.

Therefore, $\sigma_r = \sigma_\theta = \left(\frac{3+\nu}{8}\right)\rho w^2 b^2$ (6.49)

6.2.5 CIRCULAR DISK WITH A HOLE

Let a = Radius of the hole.

If there are no forces applied at the boundaries a and b, then

$(\sigma_r)_{r=a} = 0$, $(\sigma_r)_{r=b} = 0$

from which we find that

$$C = \left(\frac{3+\nu}{8}\right)\rho w^2(b^2 + a^2)$$

and $C_1 = -\left(\frac{3+\nu}{8}\right)\rho w^2 a^2 b^2$

Substituting the above in Equations (6.44) and (6.45), we obtain

$$\sigma_r = \left(\frac{3+\nu}{8}\right)\rho w^2\left(b^2 + a^2 - \left(\frac{a^2b^2}{r^2}\right) - r^2\right) \quad (6.50)$$

$$\sigma_\theta = \left(\frac{3+\nu}{8}\right)\rho w^2\left(b^2 + a^2 + \left(\frac{a^2b^2}{r^2}\right) - \left(\frac{1+3\nu}{3+\nu}\right)r^2\right) \quad (6.51)$$

The radial stress σ_r reaches its maximum at $r = \sqrt{ab}$, where

$$(\sigma_r)_{max} = \left(\frac{3+\nu}{8}\right)\rho w^2(b-a)^2 \quad (6.52)$$

The maximum circumferential stress is at the inner boundary, where

$$(\sigma_\theta)_{max} = \left(\frac{3+\nu}{4}\right)\rho w^2\left(b^2 + \left(\frac{1-\nu}{3+\nu}\right)a^2\right) \quad (6.53)$$

The displacement u_r for all the cases considered can be calculated as below:

$$u_r = r\varepsilon_\theta = \frac{r}{E}(\sigma_\theta - \nu\sigma_r) \quad (6.54)$$

6.2.6 STRESS CONCENTRATION

While discussing the case of simple tension and compression, it has been assumed that the bar has a prismatical form. Then for centrally applied forces, the stress at some distance from the ends is uniformly distributed over the cross-section. Abrupt changes in cross-section give rise to great irregularities in stress distribution. These irregularities are of particular importance in the design of machine parts subjected to variable external forces and to reversal of stresses. If there exists in the structural or machine element a discontinuity that interrupts the stress path, the stress at that discontinuity may be considerably greater than the nominal stress on the section; thus there is a "Stress Concentration" at the discontinuity. The ratio of the maximum stress to the nominal stress on the section is known as the

'Stress Concentration Factor'. Thus, the expression for the maximum normal stress in a centrically loaded member becomes

$$\sigma = K\left(\frac{P}{A}\right) \tag{6.55}$$

where A is either gross or net area (area at the reduced section), K = stress concentration factor and P is the applied load on the member. In Figures 6.8 (a), 6.8(b) and 6.8(c), the type of discontinuity is shown and in Figures 6.8(d), 6.8(e) and 6.8(f) the approximate distribution of normal stress on a transverse plane is shown.

Stress concentration is a matter, which is frequently overlooked by designers. The high stress concentration found at the edge of a hole is of great practical importance. As an example, holes in ships decks may be mentioned. When the hull of a ship is bent, tension or compression is produced in the decks and there is a high stress concentration at the holes. Under the cycles of stress produced by waves, fatigue of the metal at the overstressed portions may result finally in fatigue cracks.

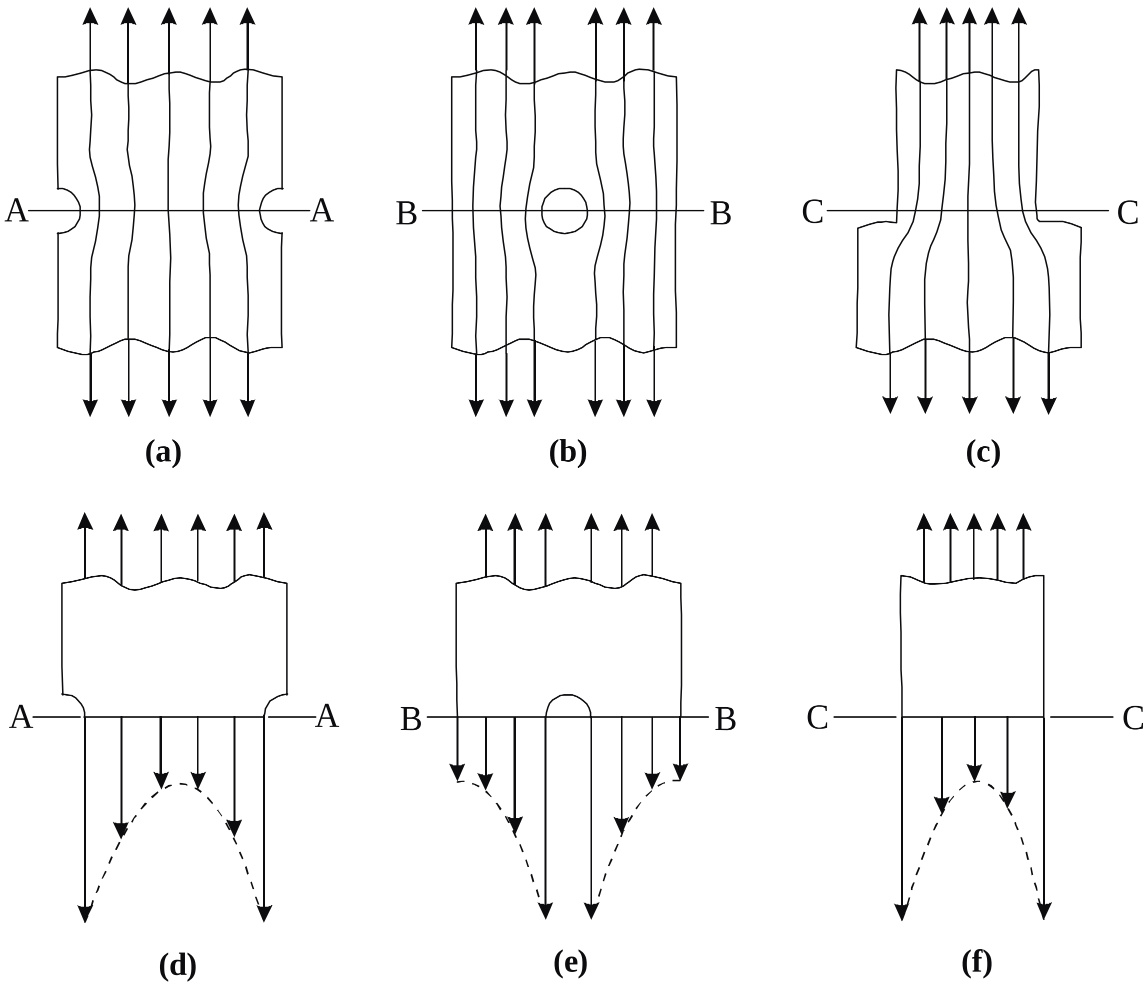


Figure 6.8 Irregularities in Stress distribution

6.2.7 THE EFFECT OF CIRCULAR HOLES ON STRESS DISTRIBUTIONS IN PLATES

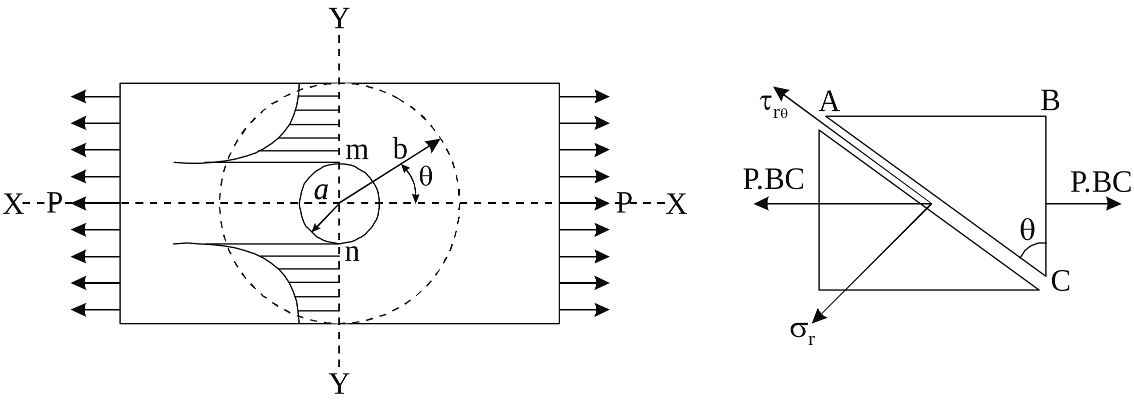


Figure 6.9 Plate with a circular hole

Consider a plate subjected to a uniform tensile stress P as shown in the Figure 6.9. The plate thickness is small in comparison to its width and length so that we can treat this problem as a plane stress case. Let a hole of radius 'a' be drilled in the middle of the plate as shown in the figure. This hole will disturb the stress field in the neighbourhood of the hole. But from St.Venant's principle, it can be assumed that any disturbance in the uniform stress field will be localized to an area within a circle of radius 'b'. Beyond this circle, it is expected that the stresses to be effectively the same as in the plate without the hole.

Now consider the equilibrium of an element ABC at $r = b$ and angle θ with respect to x-axis.

$$\therefore \sigma_r = P.BC\left(\frac{\cos\theta}{AC}\right)$$

$$= P.\cos^2\theta$$

$$\therefore \sigma_r = \frac{P}{2}(1+\cos 2\theta) \tag{6.56}$$

and $\tau_{r\theta} = -P.BC\left(\dfrac{\sin\theta}{AC}\right)$

$$= -P.\sin\theta\cos\theta$$

$$\therefore \tau_{r\theta} = -\frac{P}{2}\sin 2\theta \tag{6.57}$$

These stresses, acting around the outside of the ring having the inner and outer radii $r = a$ and $r = b$, give a stress distribution within the ring which may be regarded as consisting of two parts.

(a) A constant radial stress $\frac{P}{2}$ at radius b. This condition corresponds to the ordinary thick cylinder theory and stresses σ_r' *and* σ_θ' at radius r is given by

$$\sigma_r' = A + \left(\frac{B}{r^2}\right) \text{and } \sigma_\theta' = A - \left(\frac{B}{r^2}\right)$$

Constants A and B are given by boundary conditions,

(i) At $r = a,\ \sigma_r = 0$

(ii) At $r = b,\ \sigma_r = \frac{P}{2}$

On substitution and evaluation, we get

$$\sigma_r' = \frac{Pb^2}{2(b^2 - a^2)}\left(1 - \frac{a^2}{r^2}\right)$$

$$\sigma_\theta' = \frac{Pb^2}{2(b^2 - a^2)}\left(1 + \frac{a^2}{r^2}\right)$$

(b) The second part of the stress σ_r'' *and* σ_θ'' are functions of θ. The boundary conditions for this are:

$$\sigma_r'' = \frac{P}{2}\cos 2\theta \text{ for } r = b$$

$$\tau_{r\theta}' = -\left(\frac{P}{2}\right)\sin 2\theta \text{ for } r = b$$

These stress components may be derived from a stress function of the form,

$$\phi = f(r)\cos 2\theta$$

because with

$$\sigma_r'' = \left(\frac{1}{r^2}\right)\frac{\partial^2\phi}{\partial\theta^2} + \left(\frac{1}{r}\right)\frac{\partial\phi}{\partial r}$$

and $\sigma_\theta'' = \left(\frac{1}{r^2}\right)\frac{\partial\phi}{\partial\theta} - \left(\frac{1}{r}\right)\frac{\partial^2\phi}{\partial r\partial\theta}$

Now, the compatibility equation is given by,

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right) f(r)\cos 2\theta = 0$$

But $\frac{\partial^2}{\partial r^2} f(r)\cos 2\theta + \frac{1}{r}\frac{\partial}{\partial r} f(r)\cos 2\theta + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} f(r)\cos 2\theta$

$$= \cos 2\theta \left\{ \frac{\partial^2}{\partial r^2} f(r) + \frac{1}{r}\frac{\partial}{\partial r} f(r) - \frac{4}{r^2} f(r) \right\}$$

Therefore, the compatibility condition reduces to

$$\cos 2\theta \left\{ \frac{\partial}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{4}{r^2} \right\}^2 f(r) = 0$$

As $\cos 2\theta$ is not in general zero, we have

$$\left\{ \frac{\partial}{\partial r^2} + \frac{1}{r}.\frac{\partial}{\partial r} - \frac{4}{r^2} \right\}^2 f(r) = 0$$

We find the following ordinary differential equation to determine $f(r)$

i.e., $\left\{ \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{4}{r^2} \right\} \left\{ \frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \frac{4f}{r^2} \right\} = 0$

i.e.,

$$\frac{d^4 f}{dr^4} + \frac{1}{r}.\frac{d^3 f}{dr^3} + \frac{2}{r^3}\frac{df}{dr} - \frac{2}{r^2}\frac{d^2 f}{dr^2} - \frac{4}{r^2}\frac{d^2 f}{dr^2} + \frac{16}{r^3}\frac{df}{dr} - \frac{24f}{r^4} + \frac{1}{r}\frac{d^3 f}{dr^3} + \frac{1}{r^2}\frac{d^2 f}{dr^2} - \frac{1}{r^3}$$

$$\frac{df}{dr} - \frac{4}{r^3}\frac{df}{dr} + \frac{8f}{r^4} - \frac{4}{r^2}\frac{d^2 f}{dr^2} - \frac{4}{r^3}\frac{df}{dr} + \frac{16f}{r^4} = 0$$

or $\frac{d^4 f}{dr^4} + \frac{2}{r}\frac{d^3 f}{dr^3} - \frac{9}{r^2}\frac{d^2 f}{dr^2} + \frac{9}{r^3}\frac{df}{dr} = 0$

This is an ordinary differential equation, which can be reduced to a linear differential equation with constant co-efficients by introducing a new variable t such that $r = e^t$.

Also, $\frac{df}{dr} = \frac{df}{dt}\frac{dt}{dr} = \frac{1}{r}\frac{df}{dt}$

$$\frac{d^2 f}{dr^2} = \frac{1}{r^2}\left(\frac{d^2 f}{dt^2} - \frac{df}{dt} \right)$$

$$\frac{d^3 f}{dr^3} = \frac{1}{r^3}\left(\frac{d^3 f}{dt^3} - 3\frac{d^2 f}{dt^2} + 2\frac{df}{dt} \right)$$

$$\frac{d^4 f}{dr^4} = \frac{1}{r^4}\left(\frac{d^4 f}{dt^4} - 6\frac{d^3 f}{dt^3} + 11\frac{d^2 f}{dt^2} - 6\frac{df}{dt} \right)$$

on substitution, we get

$$\frac{1}{r^4}\left(\frac{d^4 f}{dt^4} - 6\frac{d^3 f}{dt^3} + 11\frac{d^2 f}{dt^2} - 6\frac{df}{dt} \right) + \frac{2}{r^4}\left(\frac{d^3 f}{dt^3} - 3\frac{d^2 f}{dt^2} + 2\frac{df}{dt} \right) - \frac{9}{r^4}\left(\frac{d^2 f}{dt^2} - \frac{df}{dt} \right) + \frac{9}{r^4}\left(\frac{df}{dt} \right) = 0$$

or $\frac{d^4 f}{dt^4} - 4\frac{d^3 f}{dt^3} - 4\frac{d^2 f}{dt^2} + 16\frac{df}{dt} = 0$

Let $\dfrac{df}{dt} = m$

$\therefore m^4 - 4m^3 - 4m^2 + 16m = 0$

$m^3(m-4) - 4m(m-4) = 0$

$\therefore (m-4)(m^3 - 4m) = 0$

or $m(m^2 - 4)(m-4) = 0$

$\therefore m = 0,\ m = \pm 2,\ m = 4$

$\therefore\ f(r) = Ae^{2t} + Be^{4t} + Ce^{-2t} + D$

$\therefore\ f(r) = Ar^2 + Br^4 + \dfrac{C}{r^2} + D$

The stress function may now be written in the form:

$$\phi = \left(Ar^2 + Br^4 + \frac{C}{r^2} + D\right)\cos 2\theta$$

The stress components σ_r'' *and* σ_θ'' may now expressed as

$$\sigma_r'' = \left(\frac{1}{r^2}\right)\frac{\partial^2\phi}{\partial\theta^2} + \left(\frac{1}{r}\right)\frac{\partial\phi}{\partial r}$$

$$\therefore \sigma_r'' = -\left(2A + \frac{6C}{r^4} + \frac{4D}{r^2}\right)\cos 2\theta$$

and $\sigma_\theta'' = \dfrac{\partial^2\phi}{\partial r^2}$

$$\therefore \sigma_\theta'' = \left(2A + 12Br^2 + \frac{6C}{r^4}\right)\cos 2\theta$$

and $\tau_{r\theta}'' = \left(\dfrac{1}{r^2}\right)\dfrac{\partial\phi}{\partial\theta} - \left(\dfrac{1}{r^2}\right)\dfrac{\partial^2\phi}{\partial r\partial\theta}$

$$\therefore \tau_{r\theta}'' = \left(2A + 6Br^2 - \frac{6C}{r^2} - \frac{2D}{r^2}\right)\sin 2\theta$$

The boundary conditions are,

(a) At $r = a,\ \sigma_r'' = 0$

(b) At $r = b,\ \sigma_r'' = \dfrac{P}{2}\cos 2\theta$

(c) At $r = b,\ \tau_{r\theta}'' = -\dfrac{P}{2}\sin 2\theta$

(d) At $r = a,\ \tau_{r\theta}'' = 0$

Therefore, we have on substitution in stress components,

$$2A+\frac{6C}{a^4}+\frac{4D}{a^2}=0$$

$$2A+\frac{6C}{b^4}+\frac{4D}{b^2}=-\frac{P}{2}$$

$$2A+6Ba^2-\frac{6C}{a^4}-\frac{2D}{a^2}=0$$

$$2A+6Bb^2-\frac{6C}{b^4}-\frac{2D}{b^2}=-\frac{P}{2}$$

Solving the above, we get

$$B=-\left[\frac{Pa^2b^2}{2\left(a^2-b^2\right)^3}\right]$$

If 'a' is very small in comparison to b, we may write $B\cong 0$

Now, taking approximately,

$$D=\frac{a^2P}{2}$$

$$C=-\left(\frac{a^4P}{4}\right)$$

$$A=-\left(\frac{P}{4}\right)$$

Therefore the total stress can be obtained by adding part (a) and part (b). Hence, we have

$$\sigma_r=\sigma_r'+\sigma_r''=\frac{P}{2}\left(1-\frac{a^2}{r^2}\right)+\frac{P}{2}\left(1+\frac{3a^4}{r^4}-4\frac{a^2}{r^2}\right)\cos 2\theta \tag{6.58}$$

$$\sigma_\theta=\sigma_\theta'+\sigma_\theta''=\frac{P}{2}\left(1+\frac{a^2}{r^2}\right)-\frac{P}{2}\left(1+\frac{3a^4}{r^4}\right)\cos 2\theta \tag{6.59}$$

and $$\tau_{r\theta}=\tau_{r\theta}''=-\frac{P}{2}\left(1-\frac{3a^4}{r^4}+\frac{2a^2}{r^2}\right)\sin 2\theta \tag{6.60}$$

Now, At $r=a,\ \sigma_r=0$

$$\therefore \sigma_r=P-2P\cos 2\theta$$

When $\theta=\dfrac{\pi}{2}\, or\ \dfrac{3\pi}{2}$

$$\sigma_\theta=3P$$

When $\theta = 0$ *or* $\theta = \pi$

$\sigma_\theta = -P$

Therefore, we find that at points m and n, the stress σ_θ is three times the intensity of applied stress. The peak stress 3*P* rapidly dies down as we move from $r = a$ to $r = b$ since at $\theta = \frac{\pi}{2}$

$$\sigma_\theta = \frac{P}{2}\left(2 + \frac{a^2}{r^2} + \frac{3a^4}{r^4}\right)$$

which rapidly approaches P as r increases.

From the above, one can conclude that the effect of drilling a hole in highly stressed element can lead to serious weakening.

Now, having the solution for tension or compression in one direction, the solution for tension or compression in two perpendicular directions can be obtained by superposition. However, by taking, for instance, tensile stresses in two perpendicular directions equal to p, we find at the boundary of the hole a tensile stress $\sigma_\theta = 2p.$ Also, by taking a tensile stress p in the *x*-direction and compressive stress $-p$ in the *y*-direction as shown in figure, we obtain the case of pure shear.

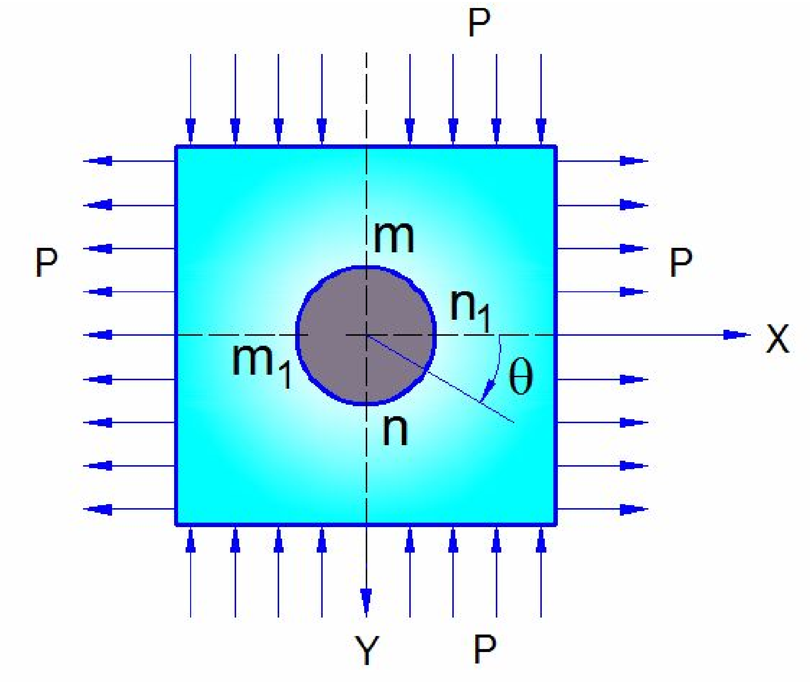


Figure 6.10 Plate subjected to stresses in two directions

Therefore, the tangential stresses at the boundary of the hole are obtained from equations (a), (b) and (c).

i.e., $\sigma_\theta = p - 2p\cos 2\theta - \left[p - 2p\cos(2\theta - \pi)\right]$

For $\theta = \frac{\pi}{2}$ *or* $\theta = \frac{3\pi}{2}$ that is, at the points n and m,

$$\sigma_\theta = 4p$$

For $\theta = 0$ *or* $\theta = \pi$, that is, at the points n_1 and m_1, $\sigma_\theta = -4p$

Hence, for a large plate under pure shear, the maximum tangential stress at the boundary of the hole is four times the applied pure shear stress.