Module: 7 Torsion of Prismatic Bars

7.1.1 INTRODUCTION

From the study of elementary strength of materials, two important expressions related to the torsion of circular bars were developed. They are

$$\tau = \frac{M_t r}{J} \tag{7.1}$$

and $$\theta = \frac{1}{L}\int_L \frac{M_t dz}{GJ} \tag{7.2}$$

Here τ represents the shear stress, M_t the applied torque, r the radius at which the stress is required, G the shear modulus, θ the angle of twist per unit longitudinal length, L the length, and z the axial co-ordinate.

Also, J = Polar moment of inertia which is defined by $\int_A r^2 dA$

The following are the assumptions associated with the elementary approach in deriving (7.1) and (7.2).

1. The material is homogeneous and obeys Hooke's Law.

2. All plane sections perpendicular to the longitudinal axis remain plane following the application of a torque, i.e., points in a given cross-sectional plane remain in that plane after twisting.

3. Subsequent to twisting, cross-sections are undistorted in their individual planes, i.e., the shearing strain varies linearly with the distance from the central axis.

4. Angle of twist per unit length is constant.

In most cases, the members that transmit torque, such as propeller shaft and torque tubes of power equipment, are circular or turbular in cross-section.

But in some cases, slender members with other than circular cross-sections are used. These are shown in the Figure 7.0.

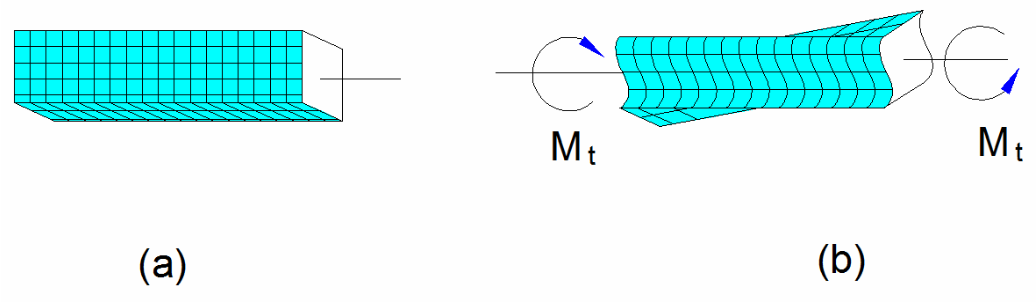

Figure 7.0 Non-Circular Sections Subjected to Torque

While treating non-circular prismatic bars, initially plane cross-sections [Figure 7.0 (a)] experience out-of-plane deformation or "Warping" [Figure 7.0(b)] and therefore assumptions 2. and 3. are no longer appropriate. Consequently, a different analytical approach is employed, using theory of elasticity.

7.1.2 GENERAL SOLUTION OF THE TORSION PROBLEM

The correct solution of the problem of torsion of bars by couples applied at the ends was given by Saint-Venant. He used the semi-inverse method. In the beginning, he made certain assumptions for the deformation of the twisted bar and showed that these assumptions could satisfy the equations of equilibrium given by

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + F_x = 0$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z} + F_y = 0$$

$$\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + F_z = 0$$

and the boundary conditions such as

$$\overline{X} = \sigma_x l + \tau_{xy} m + \tau_{xz} n$$

$$\overline{Y} = \sigma_y m + \tau_{yz} m + \tau_{xy} l$$

$$\overline{Z} = \sigma_z n + \tau_{xz} l + \tau_{yz} m$$

in which F_x, F_y, F_z are the body forces, *X, Y, Z* are the components of the surface forces per unit area and *l, m, n* are the direction cosines.

Also from the uniqueness of solutions of the elasticity equations, it follows that the torques on the ends are applied as shear stress in exactly the manner required by the solution itself.

Now, consider a prismatic bar of constant arbitrary cross-section subjected to equal and opposite twisting moments applied at the ends, as shown in the Figure 7.1(a).

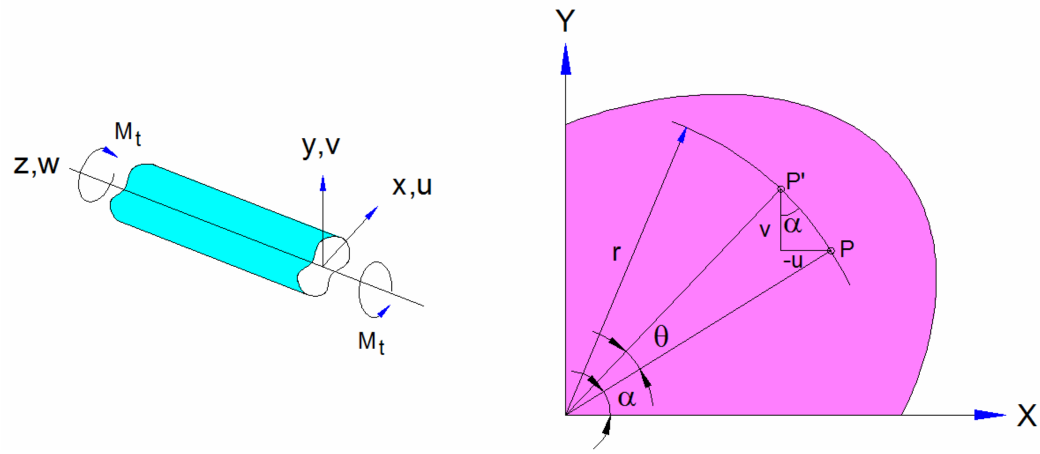

Figure 7.1 Bars subjected to torsion

Saint-Venant assumes that the deformation of the twisted shaft consists of

1. Rotations of cross-sections of the shaft as in the case of a circular shaft and
2. Warping of the cross-sections that is the same for all cross-sections.

The origin of x, y, z in the figure is located at the center of the twist of the cross-section, about which the cross-section rotates during twisting. Figure 7.1(b) shows the partial end view of the bar (and could represent any section). An arbitrary point on the cross-section, point $P(x, y)$, located a distance r from center of twist A, has moved to $P'(x\text{-}u, y+v)$ as a result of torsion. Assuming that no rotation occurs at end $z = 0$ and that θ is small, the x and y displacements of P are respectively:

$u = -(r\theta_z)\sin\alpha$

But $\sin\alpha = y / r$

Therefore, $u = -(r\theta_z)\, y/r = -y\theta_z$ (a)

Similarly, $v = (r\theta_z)\cos\alpha = (r\theta_z)\, x/r = x\theta_z$ (b)

where θ_z is the angle of rotation of the cross-section at a distance z from the origin.

The warping of cross-sections is defined by a function ψ as

$$w = \theta\,\psi\,(x, y) \tag{c}$$

Here, the equations (a) and (b) specify the rigid body rotation of any cross-section through a small angle θ_z. However, with the assumed displacements (a), (b) and (c), we calculate the components of strain from the equations given below.

$$\varepsilon_x = \frac{\partial u}{\partial x},\ \ \varepsilon_y = \frac{\partial v}{\partial y},\ \ \varepsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\ ,\ \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

and $\quad \gamma_{zx} = \dfrac{\partial w}{\partial x} + \dfrac{\partial u}{\partial z}$,

Substituting (a), (b) and (c) in the above equations, we obtain

$$\varepsilon_x = \varepsilon_y = \varepsilon_z = \gamma_{xy} = 0$$

$$\gamma_{xz} = \frac{\partial w}{\partial x} - y\,\theta = \left(\theta\frac{\partial \psi}{\partial x} - y\theta\right)$$

or $\quad \gamma_{xz} = \theta\left(\dfrac{\partial \psi}{\partial x} - y\right)$

and $\gamma_{yz} = \dfrac{\partial w}{\partial y} + x\,\theta = \left(\theta\dfrac{\partial \psi}{\partial y} + x\theta\right)$

or $\quad \gamma_{yz} = \theta\left(\dfrac{\partial \psi}{\partial y} + x\right)$

Also, by Hooke's Law, the stress-strain relationships are given by

$$\sigma_x = 2G\varepsilon_x + \lambda e\ , \quad \tau_{xy} = G\gamma_{xy}$$

$$\sigma_y = 2G\varepsilon_y + \lambda e\ , \quad \tau_{yz} = G\gamma_{yz}$$

$$\sigma_z = 2G\varepsilon_z + \lambda e\ , \quad \tau_{xz} = \gamma_{xz}$$

where $\quad e = \varepsilon_x + \varepsilon_y + \varepsilon_z$

and $\lambda = \dfrac{\nu E}{(1+\nu)(1-2\nu)}$

Substituting (a), (b) and (c) in the above equations, we obtain

$$\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0$$

$$\tau_{xz} = G\left(\frac{\partial w}{\partial x} - y\theta\right) = G\theta\left(\frac{\partial \psi}{\partial x} - y\right) \qquad \text{(d)}$$

$$\tau_{yz} = G\left(\frac{\partial w}{\partial y} + x\theta\right) = G\theta\left(\frac{\partial \psi}{\partial y} + x\right) \qquad \text{(e)}$$

It can be observed that with the assumptions (a), (b) and (c) regarding deformation, there will be no normal stresses acting between the longitudinal fibers of the shaft or in the longitudinal direction of those fibers. Also, there will be no distortion in the planes of cross-sections, since ε_x, ε_y and γ_{xy} vanish. We have at each point, pure shear defined by the components τ_{xz} and τ_{yz}.

However, the stress components should satisfy the equations of equilibrium given by:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + F_x = 0$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z} + F_y = 0 \qquad \frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + F_z = 0$$

Assuming negligible body forces, and substituting the stress components into equilibrium equations, we obtain

$$\frac{\partial \tau_{xz}}{\partial z} = 0, \qquad \frac{\partial \tau_{zy}}{\partial z} = 0, \ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} = 0 \qquad (7.3)$$

Also, the function $\psi\,(x, y)$, defining warping of cross-section must be determined by the equations of equilibrium.

Therefore, we find that the function ψ must satisfy the equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \qquad (7.3a)$$

Now, differentiating equation (d) with respect to y and the equation (e) with respect to x, and subtracting we get an equation of compatibility

Hence, $\dfrac{\partial \tau_{xz}}{\partial y} = -G\theta$

$$\frac{\partial \tau_{yz}}{\partial x} = G\theta$$

$$\frac{\partial \tau_{xz}}{\partial y} - \frac{\partial \tau_{yz}}{\partial x} = -G\theta - G\theta = -2G\theta = H$$

Therefore, $\dfrac{\partial \tau_{xz}}{\partial y} - \dfrac{\partial \tau_{yz}}{\partial x} = H$ (7.4)

Therefore the stress in a bar of arbitrary section may be determined by solving Equations (7.3) and (7.4) along with the given boundary conditions.

7.1.3 BOUNDARY CONDITIONS

Now, consider the boundary conditions given by

$$\overline{X} = \sigma_x l + \tau_{xy} m + \tau_{xz} n$$
$$\overline{Y} = \sigma_y m + \tau_{yz} n + \tau_{xy} l$$
$$\overline{Z} = \sigma_z n + \tau_{xz} l + \tau_{yz} m$$

For the lateral surface of the bar, which is free from external forces acting on the boundary and the normal n to the surface is perpendicular to the z-axis, we have $X = Y = Z = 0$ and $n = 0$. The first two equations are identically satisfied and the third gives,

$$\tau_{xz} l + \tau_{yz} m = 0 \tag{7.5}$$

which means that the resultant shearing stress at the boundary is directed along the tangent to the boundary, as shown in the Figure 7.2.

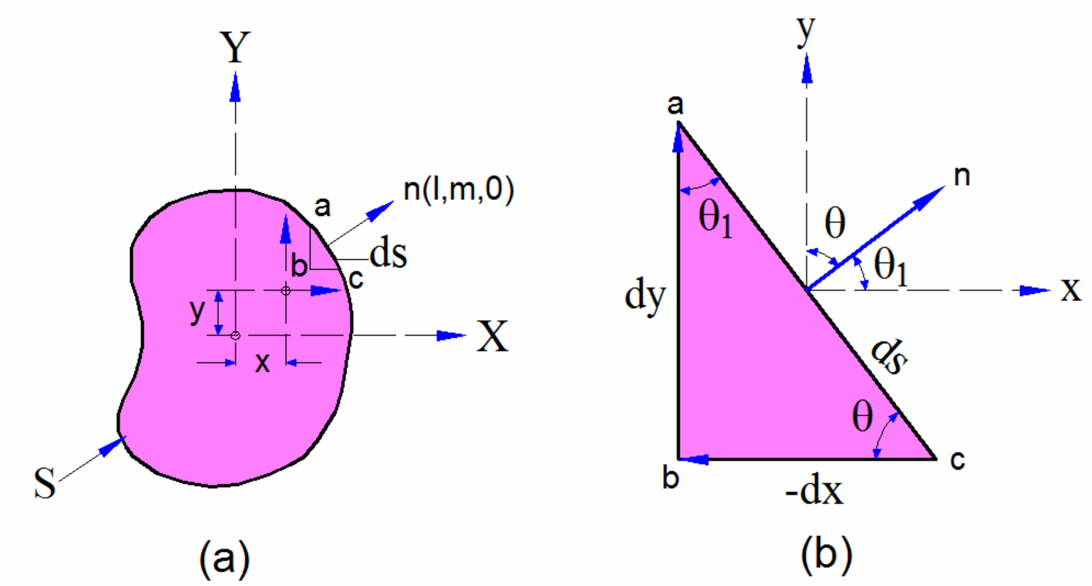

Figure 7.2 Cross-section of the bar & Boundary conditions

Considering an infinitesimal element *abc* at the boundary and assuming that *S* is increasing in the direction from *c* to *a*,

$$l = \cos(N, x) = \frac{dy}{dS}$$

$$m = \cos(N, y) = -\frac{dx}{dS}$$

$\therefore$ Equation (7.5) becomes

$$\tau_{xz}\left(\frac{dy}{dS}\right) - \tau_{yz}\left(\frac{dx}{dS}\right) = 0$$

or $$\left(\frac{\partial\psi}{\partial x} - y\right)\left(\frac{dy}{dS}\right) - \left(\frac{\partial\psi}{\partial y} + x\right)\left(\frac{dx}{dS}\right) = 0 \qquad (7.6)$$

Thus each problem of torsion is reduced to the problem of finding a function ψ satisfying equation (7.3a) and the boundary condition (7.6).

7.1.4 STRESS FUNCTION METHOD

As in the case of beams, the torsion problem formulated above is commonly solved by introducing a single stress function. This procedure has the advantage of leading to simpler boundary conditions as compared to Equation (7.6). The method is proposed by Prandtl. In this method, the principal unknowns are the stress components rather than the displacement components as in the previous approach.

Based on the result of the torsion of the circular shaft, let the non-vanishing components be τ_{zx} and τ_{yz}. The remaining stress components σ_x, σ_y and σ_z and τ_{xy} are assumed to be zero. In order to satisfy the equations of equilibrium, we should have

$$\frac{\partial\tau_{xz}}{\partial z} = 0, \quad \frac{\partial\tau_{yz}}{\partial z} = 0, \quad \frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} = 0$$

The first two are already satisfied since τ_{xz} and τ_{yz}, as given by Equations (d) and (e) are independent of z.

In order to satisfy the third condition, we assume a function $\phi(x, y)$ called Prandtl stress function such that

$$\tau_{xz} = \frac{\partial\phi}{\partial y}, \quad \tau_{yz} = -\frac{\partial\phi}{\partial x} \qquad (7.7)$$

With this stress function, (called Prandtl torsion stress function), the third condition is also satisfied. The assumed stress components, if they are to be proper elasticity solutions, have to satisfy the compatibility conditions. We can substitute these directly into the stress equations of compatibility. Alternately, we can determine the strains corresponding to the assumed stresses and then apply the strain compatibility conditions.

Therefore from Equations (7.7), (d) and (e), we have

$$\frac{\partial \phi}{\partial y} = G\theta(\frac{\partial \psi}{\partial x} - y) \qquad -\frac{\partial \phi}{\partial x} = G\theta(\frac{\partial \psi}{\partial y} + x)$$

Eliminating ψ by differentiating the first with respect to y, the second with respect to x, and subtracting from the first, we find that the stress function must satisfy the differential equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -2G\theta$$

or $$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = H \qquad (7.8)$$

where $H = -2G\theta$

The boundary condition (7.5) becomes, introducing Equation. (7.7)

$$\frac{\partial \phi}{\partial y}\frac{dy}{dS} + \frac{\partial \phi}{\partial x}\frac{dx}{dS} = \frac{d\phi}{dS} = 0 \qquad (7.9)$$

This shows that the stress function ϕ must be constant along the boundary of the cross-section. In the case of singly connected sections, example, for solid bars, this constant can be arbitrarily chosen. Since the stress components depend only on the differentials of ϕ, for a simply connected region, no loss of generality is involved in assuming $\phi = 0$ on S. However, for a multi-connected region, example shaft having holes, certain additional conditions of compatibility are imposed. Thus the determination of stress distribution over a cross-section of a twisted bar is used in finding the function ϕ that satisfies Equation (7.8) and is zero at the boundary.

Conditions at the Ends of the Twisted bar

On the two end faces, the resultants in x and y directions should vanish, and the moment about A should be equal to the applied torque M_t. The resultant in the x-direction is

$$\iint \tau_{xz}\,dxdy = \iint \frac{\partial \phi}{\partial y}dxdy = \int dx \int \frac{\partial \phi}{\partial y}dy$$

Therefore, $$\iint \tau_{xz}\,dxdy = 0 \qquad (7.10)$$

Since ϕ is constant around the boundary. Similarly, the resultant in the y-direction is

$$\iint \tau_{yz}\,dxdy = -\iint \frac{\partial \phi}{\partial x}dxdy$$

$$= -\int dy \int \frac{\partial \phi}{\partial x}dx$$

hence, $\int\int \tau_{yz} dxdy = 0$ (7.11)

Thus the resultant of the forces distributed over the ends of the bar is zero, and these forces represent a couple the magnitude of which is

$$M_t = \int\int (x\tau_{yz} - y\tau_{xz})dxdy \tag{7.12}$$

$$= -\int\int (x\frac{\partial\phi}{\partial x} + y\frac{\partial\phi}{\partial y})dxdy$$

Therefore,

$$M_t = -\int\int x\frac{\partial\phi}{\partial x}dxdy - \int\int y\frac{\partial\phi}{\partial y}dxdy$$

Integrating by parts, and observing that $\phi = 0$ at the boundary, we get

$$M_t = \int\int \phi dxdy + \int\int \phi dxdy \tag{7.13}$$

$$\therefore\ M_t = 2\int\int \phi dxdy \tag{7.14}$$

Hence, we observe that each of the integrals in Equation (7.13) contributing one half of the torque due to τ_{xz} and the other half due to τ_{yz}.

Thus all the differential equations and boundary conditions are satisfied if the stress function ϕ obeys Equations (7.8) and (7.14) and the solution obtained in this manner is the exact solution of the torsion problem.

7.1.5 TORSION OF CIRCULAR CROSS SECTION

The Laplace equation is given by

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0$$

where ψ = warping function.

The simplest solution to the above equation is

ψ = constant = C

But the boundary condition is given by the Equation (7.6) is

$$\left(\frac{\partial\psi}{\partial x} - y\right)\left(\frac{dy}{dS}\right) - \left(\frac{\partial\psi}{\partial y} + x\right)\left(\frac{dx}{dS}\right) = 0$$

Therefore, with ψ = C, the above boundary condition becomes

$(0-y)(dy/dS) - (0+x)(dx/dS) = 0$

$$- y\ \frac{dy}{dS} - x\frac{dx}{dS} = 0$$

$$or\ \frac{d}{dS}\frac{x^2 + y^2}{2} = 0$$

i.e., x^2+y^2 = constant

where (x, y) are the co-ordinates of any point on the boundary. Hence the boundary is a circle.

From Equation (c), we can write

$w = \theta\psi(x, y)$

i.e., $w = \theta C$

The polar moment of inertia for the section is

$$J = \iint (x^2 + y^2)dxdy = I_P$$

But $M_t = GI_P\theta$

or $\theta = \dfrac{M_t}{GI_P}$

Therefore, $w = \dfrac{M_t C}{GI_P}$

which is a constant. Since the fixed end has zero w at least at one point, w is zero at every cross-section (other than the rigid body displacement). Thus the cross-section does not warp.

Further, the shear stresses are given by the Equations (d) and (e) as

$$\tau_{xz} = G\left(\frac{\partial w}{\partial x} - y\theta\right) = G\theta\left(\frac{\partial \psi}{\partial x} - y\right)$$

$$\tau_{yz} = G\left(\frac{\partial w}{\partial y} + x\theta\right) = G\theta\left(\frac{\partial \psi}{\partial y} + x\right)$$

$\therefore\ \tau_{xz} = -G\theta y$

and $\tau_{yz} = G\theta x$

or $\tau_{xz} = -G\ \dfrac{M_t}{GI_P}\, y$

$$\tau_{xz} = -\ \frac{M_t x}{I_P}$$

and $\tau_{yx} = G\theta x$

$$= G\ \frac{M_t}{GI_P x}$$

hence, $\tau_{yz} = \dfrac{M_t x}{I_P}$

Therefore, the direction of the resultant shear stress τ is such that, from Figure 7.3

$$\tan\alpha = \frac{\tau_{yz}}{\tau_{xz}} = \frac{M_t x / I_P}{-M_t y / I_P} = -x/y$$

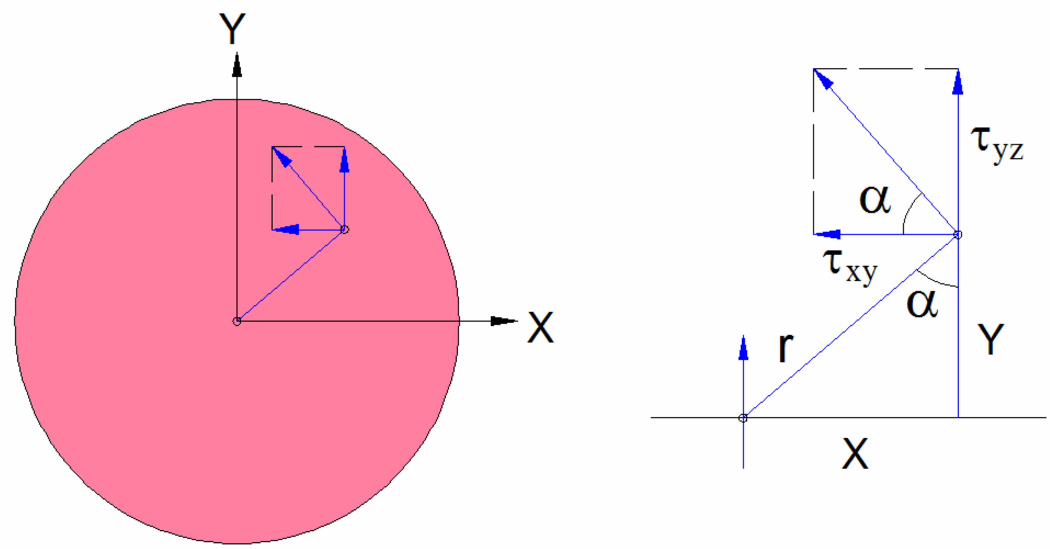

Figure 7.3 Circular bar under torsion

Hence, the resultant shear stress is perpendicular to the radius.

Further,

$$\tau^2 = \tau_{yz}^2 + \tau_{xz}^2$$

$$\tau^2 = M_t^2\,(x^2+y^2)/I_p^2$$

or $\tau = \dfrac{M_t}{I_P}\sqrt{x^2+y^2}$

Therefore, $\tau = \dfrac{M_t . r}{I_P}$

or $\tau = \dfrac{M_t . r}{J}$ (since $J = I_P$)

where r is the radial distance of the point (x, y). Hence all the results of the elementary analysis are justified.