Module 8: Elastic Solutions and Applications in Geomechanics

8.1.1 INTRODUCTION

Most of the elasticity problems in geomechanics were solved in the later part of nineteenth century and they were usually solved not for application to geotechnical pursuits, but simply to answer basic questions about elasticity and behavior of elastic bodies. With one exception, they all involve a point load. This is a finite force applied at a point: a surface of zero area. Because of stress singularities, understanding point-load problems will involve limiting procedures, which are a bit dubious in regard to soils. Of all the point-load problems, the most useful in geomechanics is the problem of a point load acting normal to the surface of an elastic half-space.

The classical problem of Boussinesq dealing with a normal force applied at the plane boundary of a semi-infinite solid has found practical application in the study of the distribution of foundation pressures, contact stresses, and in other problems of soil mechanics. Solutions of the problems of Kelvin, Flamant, Boussinesq, Cerrutti and Mindlin related to point load are discussed in the following sections.

8.1.2 KELVIN'S PROBLEM

It is the problem of a point load acting in the interior of an infinite elastic body as shown in the Figure 8.1.

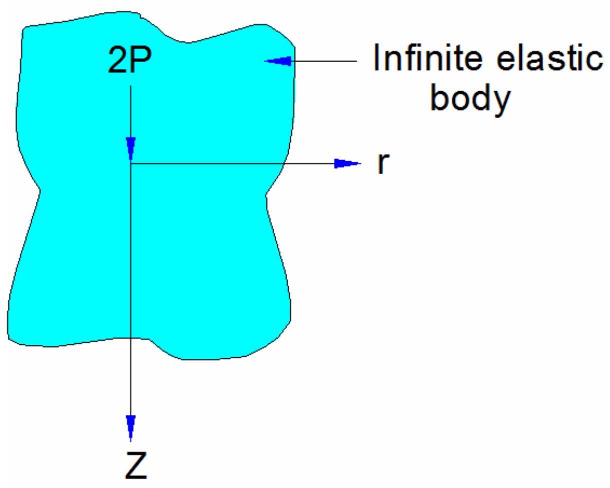

Figure 8.1 Kelvin's Problem

Consider a point load of magnitude $2P$ acting at a point in the interior of an infinite elastic body.

In the cylindrical coordinate system, the following displacements can be obtained by Kelvin's solution.

Displacement in radial direction = $u_r = \dfrac{Prz}{8\pi G(1-\nu)R^3}$

Tangential displacement = $u_\theta = 0$

Vertical displacement = $u_z = \dfrac{P}{8\pi G(1-\nu)}\left[\dfrac{2(1-2\nu)}{R} + \dfrac{1}{R} + \dfrac{z^2}{R^3}\right]$ (8.1)

Similarly, the stresses are given by

$$\sigma_r = -\frac{P}{4\pi(1-\nu)}\left[\frac{(1-2\nu)z}{R^3} - \frac{3r^2 z}{R^5}\right]$$

$$\sigma_\theta = \frac{P(1-2\nu)z}{4\pi(1-\nu)R^3}$$

$$\sigma_z = \frac{P}{4\pi(1-\nu)}\left[\frac{(1-2\nu)z}{R^3} + \frac{3z^3}{R^5}\right] \quad (8.2)$$

$$\tau_{rz} = \frac{P}{4\pi(1-\nu)}\left[\frac{(1-2\nu)r}{R^3} + \frac{3rz^2}{R^5}\right]$$

$$\tau_{r\theta} = \tau_{\theta r} = \tau_{\theta z} = \tau_{z\theta} = 0$$

Here $R = \sqrt{z^2 + r^2}$

It is clear from the above expressions that both displacements and stresses die out for larger values of R. But on the plane $z = 0$, all the stress components except for τ_{rz} vanish, at all points except the origin.

Vertical Tractions Equilibrating the applied Point Load

Consider the planar surface defined by $z = \pm h$, as shown in the Figure 8.2.

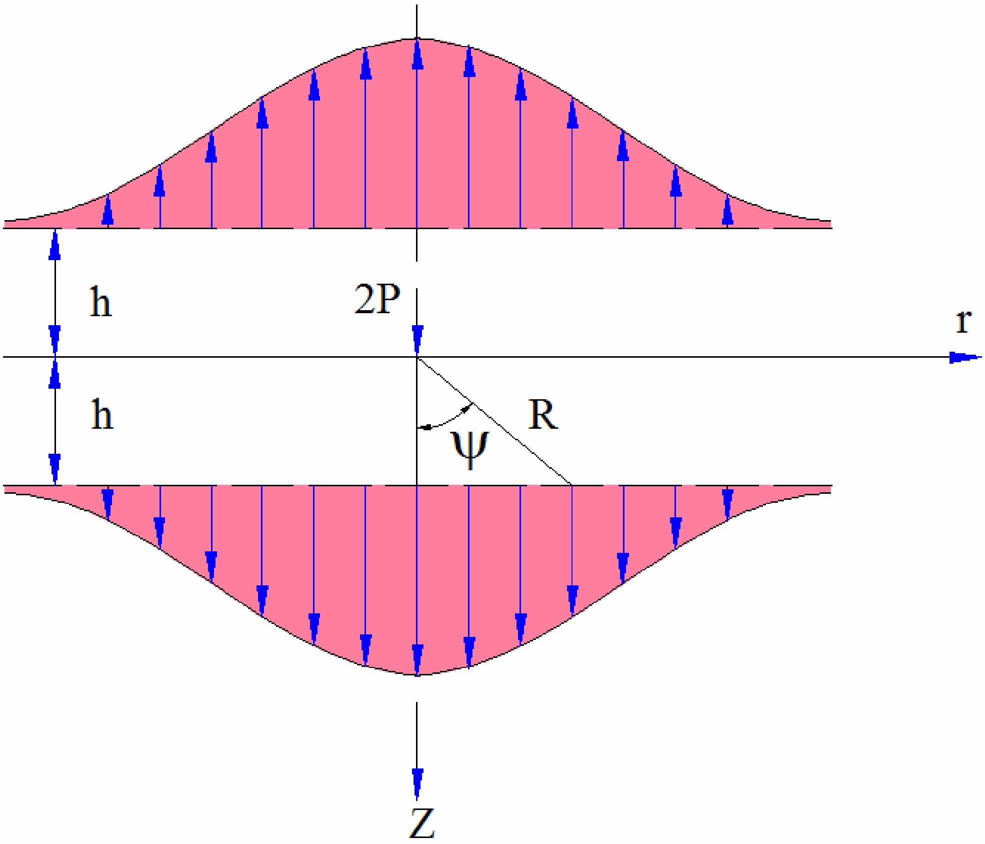

Figure 8.2 Vertical stress distributions on horizontal planes above and below point load

The vertical component of traction on this surface is σ_z. If we integrate σ_z, over this entire surface, we will get the resultant force. To find this resultant force, consider a horizontal circle centered on the *z*-axis over which σ_z is constant (Figure 8.3).

Therefore, the force acting on the annulus shown in Figure 8.3 will be $\sigma_z \times 2\pi r dr$.

Now, the total resultant force on the surface $z = h$ is given by,

Resultant upward force = $\int_o^\infty \sigma_z (2\pi r dr)$

$$= \int_o^\infty \frac{P}{4\pi(1-\nu)}\left[\frac{(1-2\nu)h}{R^3} + \frac{3h^3}{R^5}\right](2\pi r dr)$$

$$= \int_o^\infty \frac{P}{2(1-\nu)}\left[\frac{(1-2\nu)h}{R^3} + \frac{3h^3}{R^5}\right] r\, dr$$

To simplify the integration, introduce the angle ψ as shown in the Figure 8.2.

Here

$r = h\tan\psi$ and $dr = h\sec^2\psi\, d\psi$

Therefore, Resultant upward force = $\int_o^{\pi/2} \frac{P}{2(1-\nu)}[(1-2\nu)\sin\psi + 3\cos^2\psi\sin\psi]d\psi$

Solving, we get resultant upward force on the lower plane = P which is exactly one-half the applied load. Further, if we consider a similar surface $z = -h$, shown in Figure 8.2, we will find tensile stresses of the same magnitude as the compressive stresses on the lower plane.

Hence, Resultant force on the upper plane = $-P$ (tensile force). Combining the two resultant forces, we get $2P$ which exactly equilibrate the applied load.

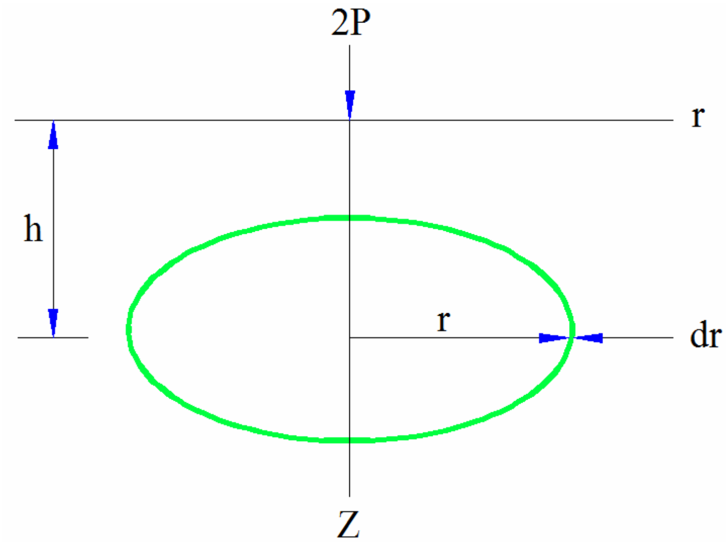

Figure 8.3 Geometry for integrating vertical stress

8.1.3 FLAMANT'S PROBLEM

Figure 8.4 shows the case of a line load of intensity *'q'* per unit length acting on the surface of a homogeneous, elastic and isotropic half-space.

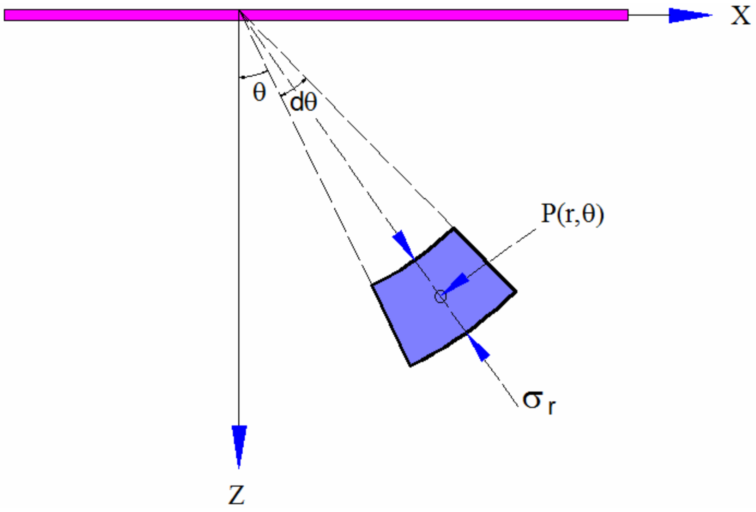

Figure 8.4 Vertical line load on Surface of an half-space

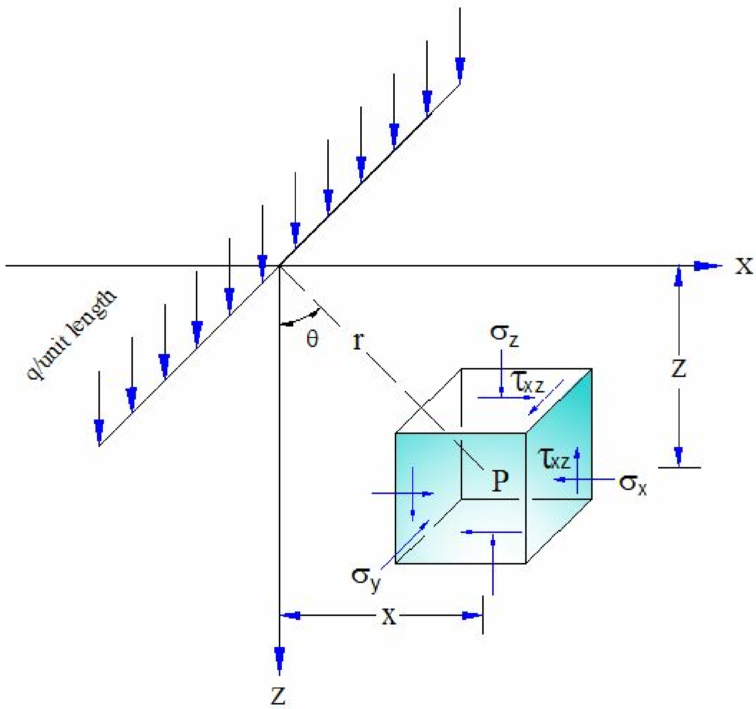

Figure 8.5 Stresses due to a vertical line load in rectangular coordinates

The stresses at a point P (r, θ) can be determined by using the stress function

$$\phi = \frac{q}{\pi} r\theta \sin\theta \tag{8.3}$$

In the polar co-ordinate system, the expressions for the stresses are as follows:

$$\sigma_r = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} \tag{8.4}$$

and $$\sigma_\theta = \frac{\partial^2\phi}{\partial r^2} \tag{8.5}$$

$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) \tag{8.6}$$

Now, differentiating Equation (8.3) with respect to r, we get

$$\frac{\partial\phi}{\partial r} = \frac{q}{\pi}\theta\sin\theta$$

Similarly $\frac{\partial^2\phi}{\partial r^2} = 0$ and $\sigma_\theta = 0$

Also, differentiating equation (8.3) with respect to θ, we get

$$\frac{\partial\phi}{\partial\theta} = \frac{q}{\pi} r[\theta\cos\theta + \sin\theta]$$

$$\frac{\partial\phi}{\partial\theta} = \frac{qr\theta\cos\theta}{\pi} + \frac{qr\sin\theta}{\pi}$$

Similarly $\frac{\partial^2\phi}{\partial\theta^2} = \frac{qr}{\pi}[\theta(-\sin\theta) + \cos\theta] + \frac{qr}{\pi}\cos\theta$

$$\frac{\partial^2\phi}{\partial\theta^2} = \frac{qr}{\pi}\cos\theta + \frac{qr}{\pi}\cos\theta - \frac{qr\theta}{\pi}\sin\theta$$

Therefore, equation (8.4) becomes

$$\sigma_r = \frac{1}{r}\left(\frac{q}{\pi}\theta\sin\theta\right) + \frac{1}{r^2}\left(\frac{q}{\pi}r\cos\theta + \frac{q}{\pi}r\cos\theta - \frac{q}{\pi}r\theta\sin\theta\right)$$

$$= \frac{q}{\pi r}\theta\sin\theta + \frac{2q}{\pi r}\cos\theta - \frac{q}{\pi r}\theta\sin\theta$$

Or $$\sigma_r = \frac{2q\cos\theta}{\pi r}$$

Now, $\frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{q}{\pi}[\theta\cos\theta + \sin\theta]$

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) = 0$$

Hence, $\tau_{r\theta} = 0$

The stress function assumed in Equation (8.3) will satisfy the compatibility equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right) = 0$$

Here σ_r and σ_θ are the major and minor principal stresses at point *P*. Now, using the above expressions for σ_r, σ_θ and $\tau_{r\theta}$, the stresses in rectangular co-ordinate system (Figure 8.5) can be derived.

Therefore,

$$\sigma_z = \sigma_r\cos^2\theta + \sigma_\theta\sin^2\theta - 2\tau_{r\theta}\sin\theta\cos\theta$$

Here, $\sigma_\theta = 0$ and $\tau_{r\theta} = 0$

Hence, $\sigma_z = \sigma_r\cos^2\theta$

$$\frac{2q}{\pi r}\cos\theta(\cos^2\theta)$$

$$\sigma_z = \frac{2q}{\pi r}\cos^3\theta$$

But from the Figure 8.5,

$$r = \sqrt{x^2 + z^2}$$

$$\cos\theta = \frac{z}{\sqrt{(x^2 + z^2)}}, \quad \sin\theta = \frac{x}{\sqrt{x^2 + z^2}}$$

Therefore,

$$\sigma_z = \frac{2q}{\pi\sqrt{(x^2 + z^2)}} \quad \frac{z^3}{\left(\sqrt{x^2 + z^2}\right)^3}$$

$$\sigma_z = \frac{2qz^3}{\pi(x^2 + z^2)^2}$$

Similarly,

$$\sigma_x = \sigma_r\sin^2\theta + \sigma_\theta\cos^2\theta + 2\tau_{r\theta}\sin\theta$$

$$= \frac{2q}{\pi r}\cos\theta\sin^2\theta + 0 + 0$$

$$= \frac{2q}{\pi\sqrt{x^2+z^2}} \frac{z}{\sqrt{x^2+z^2}} \frac{x^2}{(x^2+z^2)}$$

or $\sigma_x = \dfrac{2qx^2z}{\pi\left(x^2+z^2\right)^2}$

and $\tau_{xz} = -\sigma_\theta \sin\theta\cos\theta + \sigma_r \sin\theta\cos\theta + \tau_{r\theta}(\cos^2\theta \sin^2\theta)$

$$= 0 + \frac{2q}{\pi r}\cos\theta\sin\theta\cos\theta + 0$$

$$= \frac{2q}{\pi r}\sin\theta\cos^2\theta$$

$$= \frac{2q}{\pi r}\frac{x}{\sqrt{x^2+z^2}}\frac{z^2}{(x^2+z^2)}$$

or $\tau_{xz} = \dfrac{2qxz^2}{\pi(x^2+z^2)^2}$

But for the plane strain case,

$\sigma_y = \nu(\sigma_x + \sigma_z)$

where, ν = Poisson's ratio

Substituting the values of σ_x and σ_z in σ_y, we get

$$\sigma_y = \left[\frac{2qx^2z}{\pi(x^2+z^2)^2} + \frac{2qz^3}{\pi(x^2+z^2)^2}\right]\nu$$

$$= \frac{2q\nu z}{\pi(x^2+z^2)}[x^2+z^2]$$

or $\sigma_y = \dfrac{2q\nu z}{\pi(x^2+z^2)}$

Therefore according to Flamant's solution, the following are the stresses due to a vertical line load on the surface of an half-space.

$$\sigma_x = \frac{2qx^2z}{\pi(x^2+z^2)^2}$$

$$\sigma_y = \frac{2q\nu z}{\pi(x^2+z^2)}$$

$$\sigma_z = \frac{2qz^3}{\pi(x^2+z^2)^2} \qquad (8.7)$$

$$\tau_{xz} = \frac{2qxz^2}{\pi(x^2+z^2)^2}$$

and $\tau_{xy} = \tau_{yx} = \tau_{zy} = \tau_{yz} = 0$

8.1.4 ANALYSIS TO FIND THE TRACTIONS THAT ACT ON THE CYLINDRICAL SURFACE ALIGNED WITH LINE LOAD

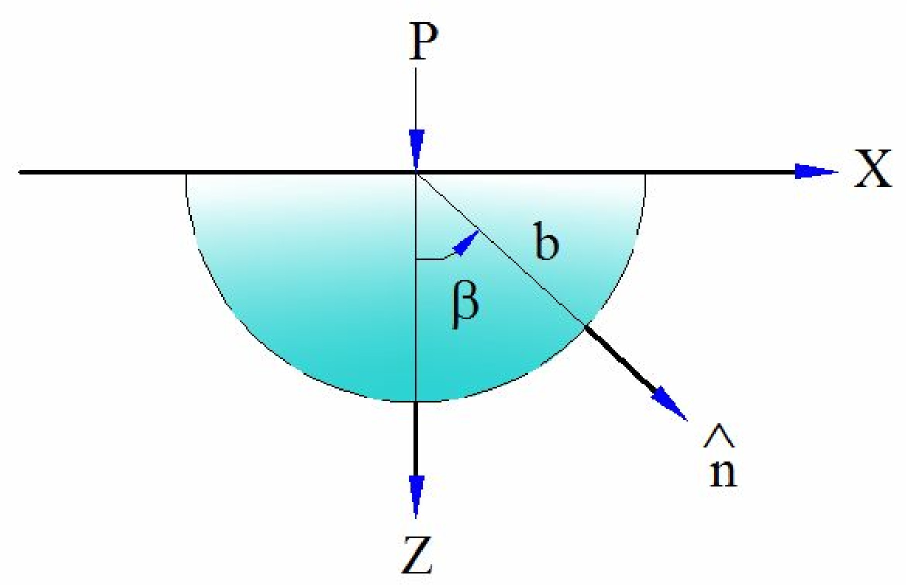

Figure 8.6 Cylindrical surface aligned with line load

One can carry out an analysis to find the tractions that act on the cylindrical surface by using the stress components in Equation (8.7).

Here, the traction vector is given by $T = \frac{2qz}{\pi b^2}\hat{n}$ (8.8)

where $\hat{n}$ is the unit normal to the cylindrical surface. This means to say that the cylindrical surface itself is a principal surface. The major principal stress acts on it.

Hence, $\sigma_1 = \frac{2qz}{\pi b^2}$ (8.9)

The intermediate principal surface is defined by $\hat{n} = \{0, 1, 0\}$ and the intermediate principal stress is $\sigma_2 = \nu\sigma_1$.

The minor principal surface is perpendicular to the cylindrical surface and to the intermediate principal surface and the minor principal stress is exactly zero.

The other interesting characteristic of Flamant's problem is the distribution of the principal stress in space.

Now, consider the locus of points on which the major principal stress σ_1 is a constant. From Equation (8.9), this will be a surface for which

$$\frac{z}{b^2} = \frac{\pi\sigma_1}{2q} = \frac{1}{2h}$$

where *C* is a constant.

But $b^2 = x^2 + z^2$

Therefore, $\dfrac{z}{(x^2+z^2)} = \dfrac{1}{2h}$

or $b^2 = (x^2+z^2) = 2hz$

which is the equation of a circle with radius *C* centered on the *z*-axis at a depth *C* beneath the origin, as shown in Figure 8.7.

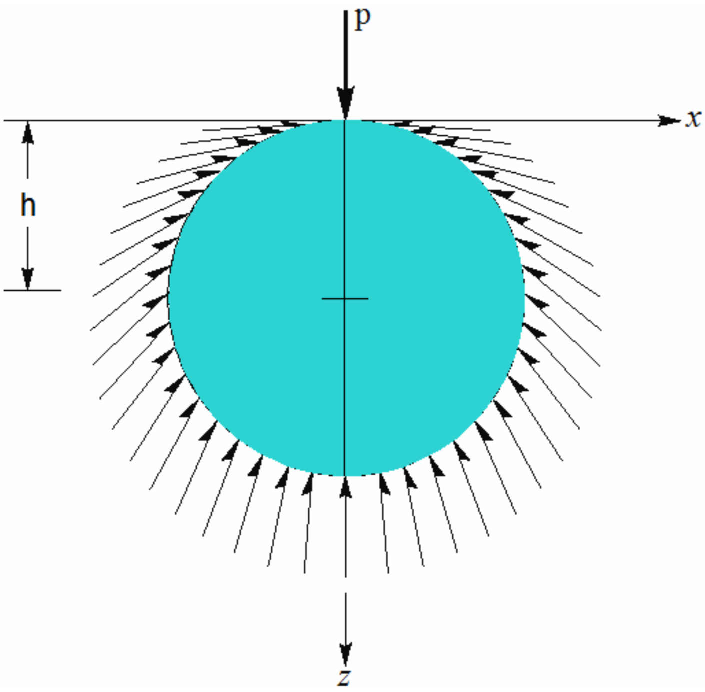

Figure 8.7 Pressure bulb on which the principal stresses are constant

At every point on the circle, the major principal stress is the same. It points directly at the origin. If a larger circle is considered, the value of σ_1 would be smaller. This result gives us the idea of a "pressure bulb" in the soil beneath a foundation.

8.1.5 BOUSSINESQ'S PROBLEM

The problem of a point load acting normal to the surface of an elastic half-space was solved by the French mathematician Joseph Boussinesq in 1878. The problem geometry is illustrated in Figure 8.8. The half-space is assumed to be homogeneous, isotropic and elastic. The point load is applied at the origin of co-ordinates on the half-space surface. Let P be the magnitude of the point load.

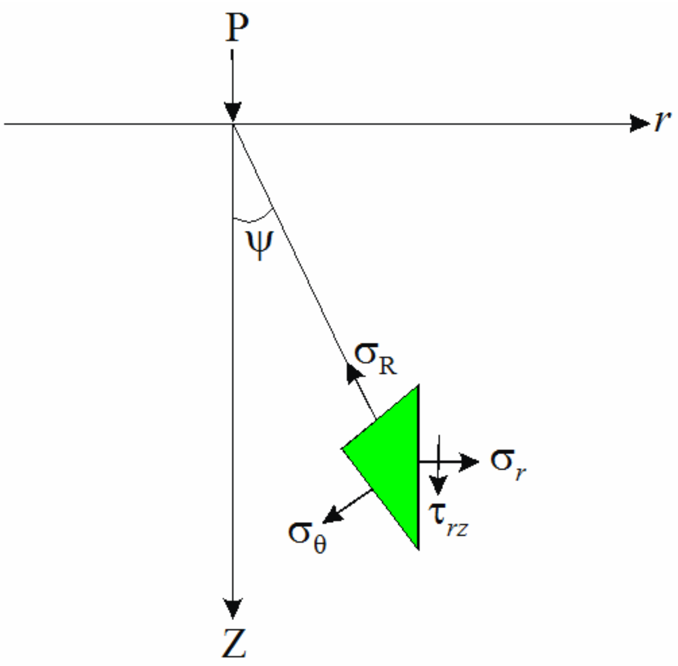

Figure 8.8 Boussinesq's problem

Consider the stress function

$$\phi = B\left(r^2 + z^2\right)^{\frac{1}{2}} \tag{8.10}$$

where B is a constant.

The stress components are given by

$$\sigma_r = \frac{\partial}{\partial z}\left(\nu\nabla^2\phi - \frac{\partial^2\phi}{\partial r^2}\right)$$

$$\sigma_\theta = \frac{\partial}{\partial z}\left(\nu\nabla^2\phi - \frac{1}{r}\frac{\partial\phi}{\partial r}\right) \tag{8.11}$$

$$\sigma_z = \frac{\partial}{\partial z}\left[(2-\nu)\nabla^2\phi - \frac{\partial^2\phi}{\partial z^2}\right]$$

$$\tau_{rz} = \frac{\partial}{\partial r}\left[(1-\nu)\nabla^2\phi - \frac{\partial^2\phi}{\partial z^2}\right]$$

Therefore, by substitution, we get

$$\sigma_r = B\left[(1-2\nu)z\left(r^2+z^2\right)^{-\frac{3}{2}} - 3r^2 z\left(r^2+z^2\right)^{-\frac{5}{2}}\right]$$

$$\sigma_\theta = B[1-2\nu]z\left(r^2+z^2\right)^{-\frac{3}{2}} \qquad (8.12)$$

$$\sigma_z - B\left[(1-2\nu)z\left(r^2+z^2\right)^{-\frac{3}{2}} + 3z^3\left(r^2+z^2\right)^{-\frac{5}{2}}\right]$$

$$\tau_{rz} = -B\left[(1-2\nu)r\left(r^2+z^2\right)^{-\frac{3}{2}} + 3rz^2\left(r^2+z^2\right)^{-\frac{5}{2}}\right]$$

Now, the shearing forces on the boundary plane $z = 0$ is given by

$$\tau_{rz} = \frac{-B(1-2\nu)}{r^2} \qquad (a)$$

In polar co-ordinates, the distribution of stress is given by

$$\sigma_R = \frac{A}{R^3},\ \sigma_\theta = \sigma_R + \frac{d\sigma_R}{dR}\frac{R}{2}$$

or $\sigma_\theta = -\frac{1}{2}\frac{A}{R^3}$

where A is a constant and $R = \sqrt{r^2+z^2}$

In cylindrical co-ordinates, we have the following expressions for the stress components:

$$\sigma_r = \sigma_R \sin^2\psi + \sigma_\theta \cos^2\psi$$

$$\sigma_z = \sigma_R \cos^2\psi + \sigma_\theta \sin^2\psi \qquad (8.13)$$

$$\tau_{rz} = \frac{1}{2}(\sigma_R - \sigma_\theta)\sin 2\psi$$

$$\sigma_\theta = -\frac{1}{2}\frac{A}{R^3}$$

But from Figure 8.8

$$\sin\psi = r\left(r^2+z^2\right)^{-\frac{1}{2}}$$

$$\cos\psi = z\left(r^2+z^2\right)^{-\frac{1}{2}}$$

Substituting the above, into σ_r, σ_z, τ_{rz} and σ_θ we get

$$\sigma_r = A\left(r^2 - \frac{1}{2}z^2\right)\left(r^2 + z^2\right)^{-\frac{5}{2}}$$

$$\sigma_z = A\left(z^2 - \frac{1}{2}r^2\right)\left(r^2 + z^2\right)^{-\frac{5}{2}}$$

$$\tau_{rz} = \frac{3}{2}\left(r^2 + z^2\right)^{-\frac{5}{2}}(A\,r\,z) \qquad (8.14)$$

$$\sigma_\theta = -\frac{1}{2}A\left(r^2 + z^2\right)^{-\frac{3}{2}}$$

Suppose now that centres of pressure are uniformly distributed along the z-axis from $z = 0$ to $z = -\infty$. Then by superposition, the stress components produced are given by

$$\sigma_r = A_1\int_z^\infty \left(r^2 - \frac{1}{2}z^2\right)\left(r^2 + z^2\right)^{-\frac{5}{2}} dz$$

$$= \frac{A_1}{2}\left[\frac{1}{r^2} - \frac{z}{r^2}(r^2 + z^2)^{-\frac{1}{2}} - z(r^2 + z^2)^{-\frac{3}{2}}\right]$$

$$\sigma_z = A_1\int_z^\infty \left(z^2 - \frac{1}{2}r^2\right)\left(r^2 + z^2\right)^{-\frac{5}{2}} dz \qquad (8.15)$$

$$= \frac{A_1}{2}z(r^2 + z^2)^{-\frac{3}{2}}$$

$$\tau_{rz} = \frac{3}{2}A_1\int_z^\infty rz\left(r^2 + z^2\right)^{-\frac{5}{2}} dz = \frac{A_1}{2}r(r^2 + z^2)^{-\frac{3}{2}}$$

$$\sigma_\theta = -\frac{1}{2}A_1\int_z^\infty \left(r^2 + z^2\right)^{-\frac{3}{2}} dz$$

$$= -\frac{A_1}{2}\left[\frac{1}{r^2} - \frac{z}{r^2}(r^2 + z^2)^{-\frac{1}{2}}\right]$$

On the plane $z = 0$, we find that the normal stress is zero and the shearing stress is

$$\tau_{rz} = \frac{1}{2}\frac{A_1}{r^2} \qquad (b)$$

From (a) and (b), it is seen that the shearing forces on the boundary plane are eliminated if,

$$-B(1 - 2v) + \frac{A_1}{2} = 0$$

Therefore, $A_1 = 2B(1 - 2v)$

Substituting the value of A_1 in Equation (8.15) and adding together the stresses (8.12) and (8.15), we get

$$\sigma_r = B\left\{(1-2\nu)\left[\frac{1}{r^2}-\frac{z}{r^2}\left(r^2+z^2\right)^{-\frac{1}{2}}\right]-3r^2z\left(r^2+z^2\right)^{-\frac{5}{2}}\right\}$$

$$\sigma_z = -3Bz^3\left(r^2+z^2\right)^{-\frac{5}{2}}$$

$$\sigma_\theta = B(1-2\nu)\left[-\frac{1}{r^2}+\frac{z}{r^2}\left(r^2+z^2\right)^{-\frac{1}{2}}+z\left(r^2+z^2\right)^{-\frac{3}{2}}\right]$$

$$\tau_{rz} = -3Brz^2\left(r^2+z^2\right)^{-\frac{5}{2}} \qquad (8.16)$$

The above stress distribution satisfies the boundary conditions, since $\sigma_z = \tau_{rz} = 0$ for $z = 0$.

To Determine the Constant B

Consider the hemispherical surface of radius *'a'* as illustrated in the Figure 8.9. For any point on this surface let $R = a =$ constant. Also, ψ be the angle between a radius of the hemisphere and the z-axis.

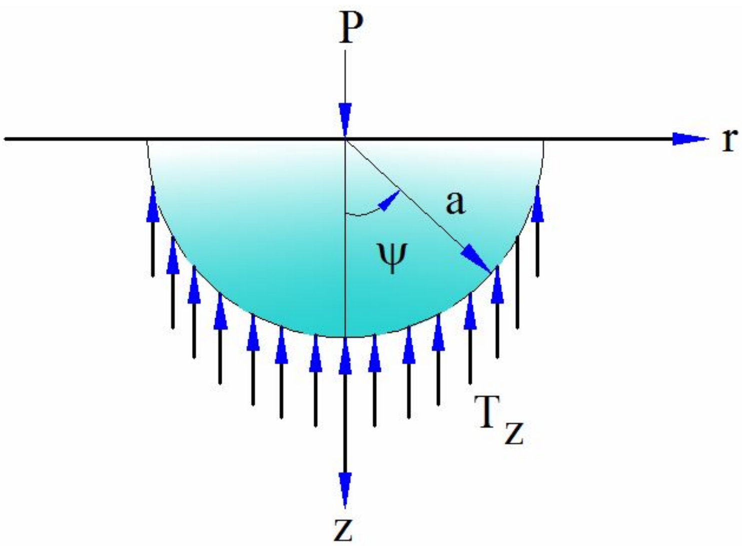

Figure 8.9 Vertical tractions acting on the hemispherical surface

The unit normal vector to the surface at any point can be written as

$$\hat{n} = \begin{bmatrix} \sin\psi \\ 0 \\ \cos\psi \end{bmatrix}$$

while r and z components of the point are

$z = a \cos \psi,\ r = a \sin \psi$

The traction vector that acts on the hemispherical surface is,

$$T = \begin{bmatrix} T_r \\ T_\theta \\ T_z \end{bmatrix} = \begin{bmatrix} \sigma_r & 0 & \tau_{rz} \\ 0 & \sigma_\theta & 0 \\ \tau_{rz} & 0 & \sigma_z \end{bmatrix} \begin{bmatrix} \sin\psi \\ 0 \\ \cos\psi \end{bmatrix} = \begin{bmatrix} \sigma_r \sin\psi + \tau_{rz} \cos\psi \\ 0 \\ \tau_{rz} \sin\psi + \sigma_z \cos\psi \end{bmatrix}$$

Considering the component of stress in the z-direction on the hemispherical surface, we have

$T_z = -(\tau_{rz} \sin\psi + \sigma_z \cos\psi)$

Substituting the values of τ_{rz}, σ_z, $\sin\psi$ and $\cos\psi$, in the above expression, we get

$T_z = 3Bz^2 (r^2 + z^2)^{-2}$

Integrating the above, we get the applied load P.

Therefore,

$$P = \int_0^{\pi/2} T_z 2\pi r \left(r^2 + z^2\right)^{\frac{1}{2}} d\psi$$

$$= \int_0^{\pi/2} \left[3Bz^2 \left(r^2 + z^2\right)^{-2}\right] \left[2\pi r \left(r^2 + z^2\right)^{\frac{1}{2}}\right] d\psi$$

$$= \int_0^{\pi/2} (6\pi B) \cos^2\psi \sin\psi d\psi$$

$$= 6\pi B \int_o^{\pi/2} \cos^2\psi \sin\psi d\psi$$

Now, solving for $\int_o^{\pi/2} \cos^2\psi \sin\psi d\psi$, we proceed as below

Put $\cos\psi = t$

i.e., $-\sin\psi,\ d\psi = dt$

If $\cos 0 = 1$, then $t = 1$

If $\cos\dfrac{\pi}{2} = 0$, then $t = 0$

Hence, $\int_0^1 -t^2 dt = -\left[\dfrac{t^3}{3}\right]_1^0 = \left[\dfrac{t^3}{3}\right]_0^1 = \dfrac{1}{3}$ Therefore, $P = 2\pi B$

Or $B = \dfrac{P}{2\pi}$

Substituting the value of *B* in Equation (8.16), we get

$$\sigma_z = -\frac{3P}{2\pi} z^3 \left(r^2 + z^2\right)^{-\frac{5}{2}}$$

$$\sigma_r = \frac{P}{2\pi}\left\{(1-2\nu)\left[\frac{1}{r^2} - \frac{z}{r^2}\left(r^2 + z^2\right)^{-\frac{1}{2}}\right] - 3r^2 z\left(r^2 + z^2\right)^{-\frac{5}{2}}\right\}$$

$$\sigma_\theta = \frac{P}{2\pi}(1-2\nu)\left\{-\frac{1}{r^2} + \frac{z}{r^2}\left(r^2 + z^2\right)^{-\frac{1}{2}} + z\left(r^2 + z^2\right)^{-\frac{3}{2}}\right\}$$

$$\tau_{rz} = -\frac{3P}{2\pi} r z^2 \left(r^2 + z^2\right)^{-\frac{5}{2}}$$

Putting $R = \sqrt{r^2 + z^2}$ and simplifying, we can write

$$\sigma_z = -\frac{3Pz^3}{2\pi R^5}$$

$$\sigma_r = \frac{P}{2\pi}\left[\frac{(1-2\nu)}{R(R+z)} - \frac{3r^2 z}{R^5}\right]$$

$$\sigma_\theta = \frac{P(1-2\nu)}{2\pi}\left[\frac{z}{R^3} - \frac{1}{R(R+z)}\right]$$

$$\tau_{rz} = -\frac{3P}{2\pi}\frac{rz^2}{R^5} \qquad (8.16a)$$

Also, Boussinesq found the following displacements for this case of loading.

$$u_r = \frac{P}{4\pi GR}\left[\frac{rz}{R^2} - \frac{(1-2\nu)r}{R+z}\right]$$

$$u_\theta = 0 \qquad (8.16b)$$

$$u_z = \frac{P}{4\pi GR}\left[2(1-\nu) + \frac{z^2}{R^2}\right]$$

8.1.6 COMPARISON BETWEEN KELVIN'S AND BOUSSINESQ'S SOLUTIONS

On the plane $z = 0$, all the stresses given by Kelvin vanish except τ_{rz}. For the special case where Poisson's ratio $\nu = 1/2$ (an incompressible material), then τ_{rz} will also be zero on this surface, and that part of the body below the $z = 0$ plane becomes equivalent to the half space of Boussinesq's problem. Comparing Kelvin's solution (with ν = 1/2) with Boussinesq's solution (with ν = 1/2), it is clear that for all $z \geq 0$, the solutions are identical. For $z \leq 0$, we also have Boussinesq's solution, but with a negative load $-P$. The two half-spaces, which together comprise the infinite body of Kelvin's problem, act as if they are uncoupled on the plane $z = 0$, where they meet.

Further, a spherical surface is centered on the origin, we find a principal surface on which the major principal stress is acting. The magnitude of the principal stress is given by

$$\sigma_1 = \frac{3Pz}{2\pi R^3} \tag{8.17}$$

where R is the sphere radius. It can be observed that the value of σ_1 changes for negative values of z, giving tensile stresses above the median plane $z = 0$.

8.1.7 CERRUTTI'S PROBLEM

Figure 8.10 shows a horizontal point load P acting on the surface of a semi-infinite soil mass.

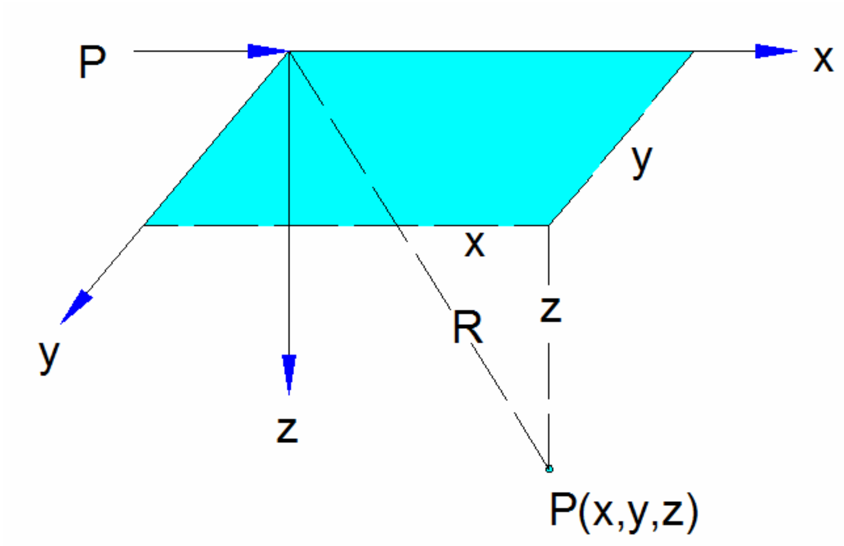

Figure 8.10 Cerrutti's Problem

The point load represented by P acts at the origin of co-ordinates, pointing in the x-direction. This is a more complicated problem than either Boussinesq's or Kelvin problem due to the absence of radial symmetry. Due to this a rectangular co-ordinate system is used in the solution.

According to Cerrutti's solution, the displacements are given by

$$u_x = \frac{P}{4\pi GR}\left\{1 + \frac{x^2}{R^2} + (1-2\nu)\left[\frac{R}{R+z} - \frac{x^2}{(R+z)^2}\right]\right\} \tag{8.18}$$

$$u_y = \frac{P}{4\pi GR}\left\{\frac{xy}{R^2} - (1-2\nu)\frac{xy}{(R+z)^2}\right\} \tag{8.18a}$$

$$u_z = \frac{P}{4\pi GR}\left\{\frac{xz}{R^2} + (1-2\nu)\frac{x}{R+z}\right\} \tag{8.18b}$$

and the stresses are

$$\sigma_x = -\frac{Px}{2\pi R^3}\left\{-\frac{3x^2}{R^2} + \frac{(1-2\nu)}{(R+z)^2}\left[R^2 - y^2 - \frac{2Ry^2}{(R+z)}\right]\right\} \tag{8.19}$$

$$\sigma_y = -\frac{Px}{2\pi R^3}\left\{-\frac{3y^2}{R^2} + \frac{(1-2\nu)}{(R+z)^2}\left[3R^2 - x^2 - \frac{2Rx^2}{(R+z)}\right]\right\} \tag{8.19a}$$

$$\sigma_z = \frac{3Pxz^2}{2\pi R^5} \tag{8.19b}$$

$$\tau_{xy} = -\frac{Py}{2\pi R^3}\left\{-\frac{3x^2}{R^2} + \frac{(1-2\nu)}{(R+z)^2}\left[-R^2 + x^2 + \frac{2Rx^2}{(R+z)}\right]\right\} \tag{8.19c}$$

$$\tau_{yz} = \frac{3Pxyz}{2\pi R^5} \tag{8.19d}$$

$$\tau_{zx} = \frac{3Px^2 z}{2\pi R^5} \tag{8.19e}$$

Here, $R^2 = x^2+y^2+z^2$

It is observed from the above stress components that the stresses approach to zero for large value of R. Inspecting at the x-component of the displacement field, it is observed that the particles are displaced in the direction of the point load. The y-component of displacement moves particles away from the x-axis for positive values of x and towards the x-axis for negative x. The plot of horizontal displacement vectors at the surface $z = 0$ is shown in Figure 8.11 for the special case of an incompressible material. Vertical displacements take the sign of x and hence particles move downward in front of the load and upward behind the load.

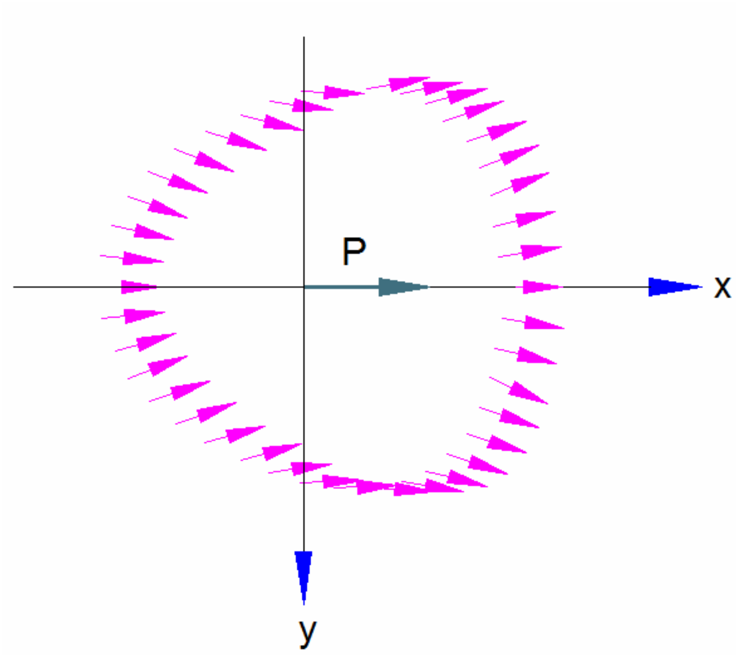

Figure 8.11 Distribution of horizontal displacements surrounding the point load

8.1.8 MINDLIN'S PROBLEM

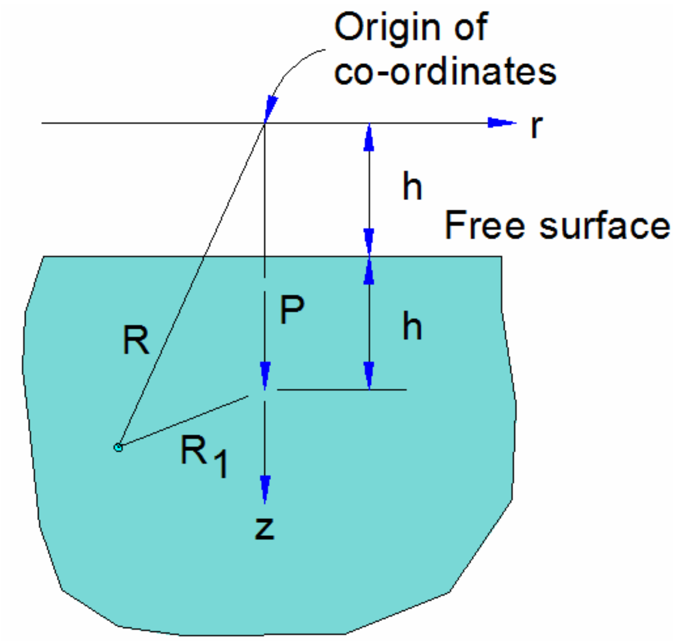

Figure 8.12 Mindlin's Problem

The two variations of the point-load problem were solved by Mindlin in 1936. These are the problems of a point load (either vertical or horizontal) acting in the interior of an elastic half space. Mindlin's problem is illustrated in Figure 8.12. The load P acts at a point located a distance z beneath the half-space surface. Such problems are more complex than Boussinesq's or Kelvin or Cerrutti's. They have found applications in the computations of the stress and displacement fields surrounding an axially loaded pile and also in the study of interaction between foundations and ground anchors.

It is appropriate to write Mindlin's solution by placing the origin of co-ordinates a distance C above the free surface as shown in the Figure 8.12. Then the applied load acts at the point z = 2h.

From Figure 8.12,

$R^2 = r^2 + z^2$

$R_1^2 = r^2 + z_1^2$

where $z_1 = z - 2h$

Here z_1 and R_1 are the vertical distance and the radial distance from the point load.

For the vertical point load, Mindlin's solution is most conveniently stated in terms of Boussinesq's solution. For example, consider the displacement and stress fields in Boussinesq's problem in the region of the half-space below the surface $z = C$. These displacements and stresses are also found in Mindlin's solution, but with additional terms. The following equations will give these additional terms.

To obtain the complete solution, add them to Equations (8.7a) and (8.7b)

Therefore,

$$\sigma_r = \frac{P}{8\pi(1-\nu)}\left\{\frac{3r^2 z_1}{R_1^5} - \frac{(1-2\nu)z_1}{R_1^3} + \frac{(1-2\nu)z - 12(1-\nu)c}{R^3} - \frac{3r^2 z - 6(7-2\nu)cz^2 + 24c^2 z}{R^5} - \frac{30cz^2(z-c)}{R^7}\right\} \tag{8.20}$$

$$\sigma_\theta = \frac{P}{8\pi(1-\nu)}\left\{-\frac{(1-2\nu)z_1}{R_1^3}\right\} + \frac{(1-2\nu)(z+6c)}{R^3} - \frac{6(1-2\nu)cz^2 - 6c^2 z}{R^5} \tag{8.20a}$$

$$\sigma_z = \frac{P}{8\pi(1-\nu)}\left\{\frac{3z_1^3}{R_1^5} + \frac{(1-2\nu)z_1}{R_1^3} - \frac{(1-2\nu)(z-2c)}{R^3} - \frac{3z^3 + 12(2-\nu)cz^2 - 18c^2 z}{R^5} + \frac{30cz^2(z-c)}{R^7}\right\} \tag{8.20b}$$

$$\tau_{rz} = \frac{Pr}{8\pi(1-\nu)}\left\{\frac{3z_1^2}{R_1^5} + \frac{(1-2\nu)}{R_1^3} - \frac{(1-2\nu)}{R^3} - \frac{3z^2 + 6(3-2\nu)cz - 6c^2}{R^5} + \frac{30cz^2(z-c)}{R^7}\right\} \tag{8.20c}$$

and $\tau_{r\theta} = \tau_{\theta r} = \tau_{\theta z} = \tau_{z\theta} = 0$ (8.20d)

Mindlin's solution for a horizontal point load also employs the definitions for z_1 and R_1. Now introduce rectangular coordinate system because of the absence of cylindrical symmetry.

Replace r^2 by x^2+y^2, and assume (without any loss of generality) that the load acts in the x-direction at the point $z = c$. Here the solution is conveniently stated in terms of Cerrutti's solution, just as the vertical point load was given in terms of Boussinesq's solution. Therefore, the displacements and stresses to be superposed on Cerrutti's solution are

$$u_x = \frac{P}{16\pi(1-\nu)G}\left\{\frac{x^2}{R_1^3} + \frac{(3-4\nu)}{R_1} - \frac{(3-4\nu)}{R} + \frac{\left(-x^2 + 2c(z-c)\right)}{R^3} - \frac{\left(6cx^2(z-c)\right)}{R^5}\right\} \tag{8.21}$$

$$u_y = \frac{P}{16\pi(1-\nu)G}\left\{\frac{xy}{R_1^3} - \frac{xy}{R^3} - \frac{6cxy(z-c)}{R^5}\right\} \tag{8.21a}$$

$$u_z = \frac{P}{16\pi(1-\nu)G}\left\{\frac{xz_1}{R_1^3} - \frac{xz + 2(3-4\nu)cz}{R^3} - \frac{6cxz(z-c)}{R^5}\right\} \tag{8.21b}$$

$$\sigma_x = \frac{Px}{8\pi(1-\nu)}\left\{\frac{3x^2}{R_1^5} + \frac{(1-2\nu)}{R_1^3} - \frac{(1-2\nu)}{R^3} - \frac{3x^2 - 6(3-2\nu)cz + 18c^2}{R^5} - \frac{30cx^2(z-c)}{R^7}\right\} \tag{8.21c}$$

$$\sigma_y = \frac{Px}{8\pi(1-\nu)}\left\{\frac{3y^2}{R_1^5} - \frac{(1-2\nu)}{R_1^3} + \frac{(1-2\nu)}{R^3} - \frac{3y^2 - 6(1-2\nu)cz + 6c^2}{R^5} - \frac{30cy^2(z-c)}{R^7}\right\} \tag{8.21d}$$

$$\sigma_z = \frac{Px}{8\pi(1-\nu)}\left\{\frac{3z_1^2}{R_1^5} - \frac{(1-2\nu)}{R_1^3} + \frac{(1-2\nu)}{R^3} - \frac{3z^2 + 6(1-2\nu)cz + 6c^2}{R^5} - \frac{30cz^2(z-c)}{R^7}\right\} \tag{8.21e}$$

$$\tau_{xy} = \frac{Py}{8\pi(1-\nu)}\left\{\frac{3x^2}{R_1^5} + \frac{1-2\nu}{R_1^3} - \frac{1-2\nu}{R^3} - \frac{3x^2 - 6c(z-c)}{R^5} - \frac{30cx^2(z-c)}{R^7}\right\} \tag{8.21f}$$

$$\tau_{yz} = \frac{Pxy}{8\pi(1-\nu)}\left\{\frac{3z_1}{R_1^5} - \frac{3z + 6(1-2\nu)c}{R^5} - \frac{30cz(z-c)}{R^7}\right\} \tag{8.21g}$$

$$\tau_{zx} = \frac{P}{8\pi(1-\nu)}\left\{\frac{3x^2z_1}{R_1^5} + \frac{(1-2\nu)z_1}{R_1^3} - \frac{(1-2\nu)(z-2c)}{R^3} - \frac{3x^2z + 6(1-2\nu)cx^2 - 6cz(z-c)}{R^5} - \frac{30cx^2z(z-c)}{R^7}\right\} \tag{8.21h}$$

8.1.9 APPLICATIONS

The mechanical response of naturally occurring soils are influenced by a variety of factors. These include (i) the shape, size and mechanical properties of the individual soil particles, (ii) the configuration of the soil structure, (iii) the intergranular stresses and stress history, and (iv) the presence of soil moisture, the degree of saturation and the soil permeability. These factors generally contribute to stress-strain phenomena, which display markedly non-linear, irreversible and time dependent characteristics, and to soil masses, which exhibit anisotropic and non-homogeneous material properties. Thus, any attempt to solve a soil-foundation interaction problem, taking into account all such material characteristics, is clearly a difficult task. In order to obtain meaningful and reliable information for practical problems of soil-foundation interaction, it becomes necessary to idealise the behaviour of soil by taking into account specific aspects of its behavior. The simplest type of idealised soil response assumes linear elastic behaviour of the supporting soil medium.

In general, one can divide the foundation problems into two classes, (1) interactive problems and (2) noninteractive problems. In the case of interactive problems, the elasticity of the foundation plays an important role. For example, a flexible raft foundation supporting a multistorey structure, like that illustrated in Figure 8.13 interacts with the soil. In terms of elasticity and structural mechanics, the deformation of the raft and the deformation of the soil must both obey requirements of equilibrium and must also be geometrically compatible. If a point on the raft is displaced relative to another point, then it can be realised that the bending stresses will develop within the raft and there will be different reactive pressures in the soil beneath those points. The response of the raft and the response of the soil are coupled and must be considered together.

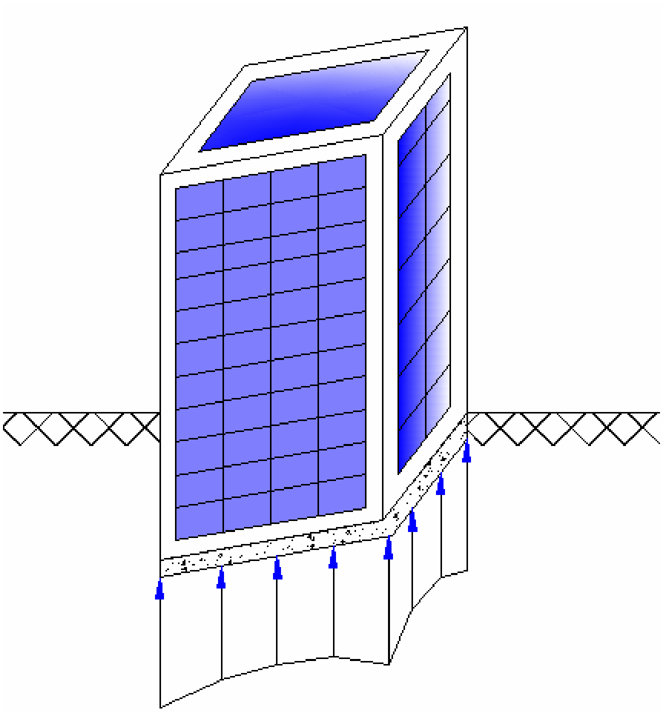

Figure 8.13 Flexible raft foundation supporting a multistorey structure

But non-interactive problems are those where one can reasonably assume the elasticity of the foundation itself is unimportant to the overall response of the soil. Examples of non-interactive problems are illustrated in Figure 8.14.

The non-interactive problems are the situations where the structural foundation is either very flexible or very rigid when compared with the soil elasticity. In non-interactive problems, it is not necessary to consider the stress-strain response of the foundation. The soil deformations are controlled by the contact exerted by the foundation, but the response of the soil and the structure are effectively uncoupled.

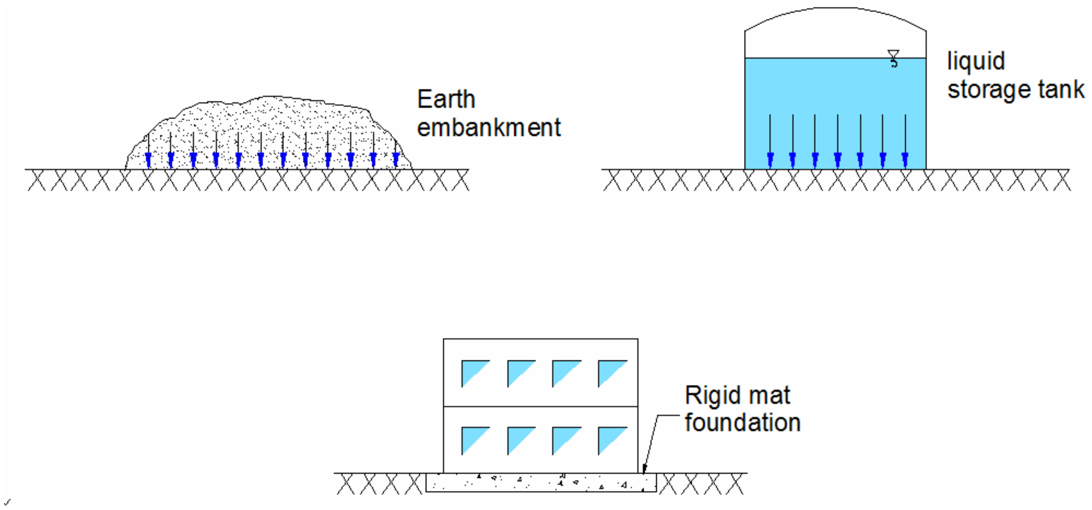

Figure 8.14 Examples of non-interactive problems

Therefore it is clear that non-interactive problems will be much simpler than interactive problems. Because of their convenience and simplicity, non-interactive problems such as a uniform vertical stress applied at the surface of a homogeneous, isotropic, elastic half-space are considered here. Generally, one can determine some of the stresses and displacements just by integrating Boussinesq's fundamental equations over the region covered by the load. There are other methods for finding appropriate solutions to these types of foundation problems, in which specialized mathematical models have been developed for soils that mimic some of the response characteristics of linear elasticity. One of the simplest model is the "Winkler model".