1- DEFORMATION PLANE

Cas d'un ouvrage de dimension suivant l'axe Oz très importante par rapport à celle de Ox et Oy,

Le tenseur de déformation

$$\begin{cases} \varepsilon_x = f_1(x,y) \\ \varepsilon_y = f_2(x,y) \\ \gamma_{xy} = f_3(x,y) \\ \varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0 \end{cases} \qquad [\varepsilon] = \begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & 0 \\ \frac{1}{2}\gamma_{xy} & \varepsilon_y & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Le vecteur de déplacement

$$\begin{cases} U = f_4(x,y) \\ V = f_5(x,y) \\ W = 0 \end{cases}$$

Pour déterminer de tenseur de contrainte.

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - v(\sigma_x + \sigma_y)\right] = 0 \Longrightarrow \sigma_z = v(\sigma_x + \sigma_y)$$

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - v(\sigma_y + \sigma_z)\right] = \frac{1}{E}\left[\sigma_x - v(\sigma_y) + v(\sigma_x + \sigma_y)\right]$$

$$\varepsilon_x = \frac{1}{E}\left[(1 - v^2)\sigma_x - v(1 + v)\sigma_y\right] = \frac{1 - v^2}{E}\left[\sigma_x - \frac{v}{1 - v}\sigma_y\right]$$

$E_1 = \frac{E}{1-v^2}$: Module d'élasticité réduit.

$v_1 = \frac{v}{1-v}$: Coefficient de poisson réduit.

D'où
$$\begin{cases} \varepsilon_x = \frac{1}{E_1}(\sigma_x - v_1\sigma_y) \\ \varepsilon_y = \frac{1}{E_1}(\sigma_y - v_1\sigma_x) \\ \gamma_{xy} = \frac{\tau_{xy}}{G} \end{cases}$$

$$[\sigma] = \begin{bmatrix} \sigma_x & \tau_{xy} & 0 \\ \tau_{xy} & \sigma_y & 0 \\ 0 & 0 & v(\sigma_x + \sigma_y) \end{bmatrix}$$

a-équations de l'étude statique (équations de Navier)

**équations différentielles d'équilibre*

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X = 0$$
$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + Y = 0$$

**équations aux conditions limites*

$$\bar{X} = l\sigma_x + m\tau_{xy}$$
$$\bar{Y} = l\tau_{xy} + m\sigma_y$$

b-équations de l'étude géométrique

**équations de CAUCHY*

$$\begin{cases} \varepsilon_x = \dfrac{\partial u}{\partial x} \\ \varepsilon_y = \dfrac{\partial v}{\partial y} \\ \gamma_{xy} = \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{cases}$$

**équations de compatibilité (*St Venant*)*

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

c- équations de l'étude physique

**lois de Hooke*

$$\varepsilon_x = \frac{1}{E_1}\left(\sigma_x - v_1\sigma_y\right)$$
$$\varepsilon_y = \frac{1}{E_1}\left(\sigma_y - v_1\sigma_x\right)$$
$$\gamma_{xy} = \frac{\tau_{xy}}{G}\,;\, \varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$$

**équations de LAME*

$$\sigma_x = (2G + \lambda)\varepsilon_x + \lambda\varepsilon_y = \lambda\varepsilon v_0 + 2G\varepsilon_x$$
$$\sigma_y = (2G + \lambda)\varepsilon_y + \lambda\varepsilon_x = \lambda\varepsilon v_0 + 2G\varepsilon_y$$
$$\sigma_z = \lambda\left(\varepsilon_x + \varepsilon_y\right) = \lambda\varepsilon v_0$$
$$\tau_{xy} = G\gamma_{xy}\,;\, \tau_{yz} = \tau_{xz} = 0$$
$$\varepsilon v_0 = \varepsilon_x + \varepsilon_y$$

2-CONTRAINTE PLANE (ELASTICITE DES TRANCHES MINCES)

Contrainte ou déformation plane

Lors d'une dimension négligeable devant les autres (Ɛ plane)

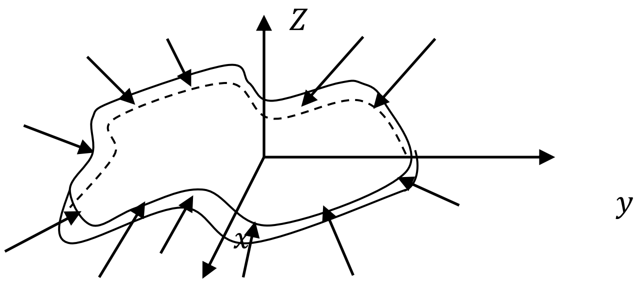
Exemple : plaque mince, membrane, dalle plancher, etc

Lors d'une dimension négligeable devant les autres (σ plane)

Exemple :…barrage, tunnel, poutre, poteau, etc

$$\Longrightarrow \sigma_z = \tau_{xz} = \tau_{yz} = 0$$

$$\sigma_x = f_1(x, y)$$
$$\sigma_y = f_2(x, y)$$
$$\tau_{xy} = f_3(x, y)$$



$$[\sigma] = \begin{bmatrix} \sigma_x & \tau_{xy} & 0 \\ \tau_{xy} & \sigma_y & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\sigma_z = 0 \implies$

$\varepsilon_x = \frac{1}{E}\left[\sigma_x - v(\sigma_y + 0)\right] = \frac{1}{E}\left[\sigma_x - v\sigma_y\right]$

$\varepsilon_z = -\frac{v}{E}\left[\sigma_x + \sigma_y\right]$; $\varepsilon_y = \frac{1}{E}\left[\sigma_y - v\sigma_x\right]$

$\gamma_{xy} = \frac{\tau_{xy}}{G}$; $\gamma_{xz} = \gamma_{yz} = 0$

$$[\varepsilon] = \begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & 0 \\ \frac{1}{2}\gamma_{xy} & \varepsilon_y & 0 \\ 0 & 0 & -\frac{v}{E}(\sigma_x + \sigma_y) \end{bmatrix}$$

3-<u>EQUATIONS D'ELASTICITE PLANE EN DEPLACEMENT (LAME)</u>

<u>a- Cas de la déformation Plane</u>

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + Y = 0$$

$$\begin{cases} \varepsilon_x = \frac{1}{E}\llbracket(1 - v^2)\sigma_x - v(1 + v)\sigma_y \\ \varepsilon_y = \frac{1}{E}\llbracket(1 - v^2)\sigma_y - v(1 + v)\sigma_x \\ \gamma_{xy} = \frac{\tau_{xy}}{G} \end{cases}$$

$$\begin{cases} \sigma_x = 2G\varepsilon_x + \lambda\varepsilon v_0 \\ \varepsilon v_0 = \sigma_x + \sigma_y \; ; \; \varepsilon_z = 0 \end{cases}$$

$$\begin{cases} \sigma_x = (2G+\lambda)\varepsilon_x + \lambda\varepsilon_y \\ \sigma_y = (2G+\lambda)\varepsilon_y + \lambda\varepsilon_x \\ \tau_{xy} = G\gamma_{xy} \end{cases}$$

$$\begin{cases} (2G + \lambda)\frac{\partial \varepsilon_x}{\partial x} + \lambda\frac{\partial \varepsilon_y}{\partial x} + G\frac{\partial \gamma_{xy}}{\partial y} + X = 0 \\ G\frac{\partial \gamma_{xy}}{\partial x} + (2G + \lambda)\frac{\partial \varepsilon_y}{\partial y} + \lambda\frac{\partial \varepsilon_x}{\partial y} + Y = 0 \end{cases}$$

$$\varepsilon_x = \frac{\partial u}{\partial x} \; ; \; \varepsilon_y = \frac{\partial v}{\partial y} \; ; \; \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\begin{cases}(2G+\lambda)\dfrac{\partial^2 u}{\partial x^2}+\lambda\dfrac{\partial^2 v}{\partial x\partial y}+G\left(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 v}{\partial x\partial y}\right)+X=0\\ G\left(\dfrac{\partial^2 u}{\partial x\partial y}+\dfrac{\partial^2 v}{\partial x^2}\right)+(2G+\lambda)\dfrac{\partial^2 v}{\partial y^2}+\lambda\dfrac{\partial^2 u}{\partial x\partial y}+Y=0\end{cases}$$

$$\begin{cases}G\left(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}\right)+(G+\lambda)\left(\dfrac{\partial^2 u}{\partial x^2}+\lambda\dfrac{\partial^2 v}{\partial x\partial y}\right)+X=0\\ G\left(\dfrac{\partial^2 v}{\partial x^2}+\dfrac{\partial^2 v}{\partial y^2}\right)+(G+\lambda)\left(\dfrac{\partial^2 u}{\partial x\partial y}+\lambda\dfrac{\partial^2 v}{\partial y^2}\right)+Y=0\end{cases}$$

$$\begin{cases}G\nabla^2 u+(G+\lambda)\dfrac{\partial\varepsilon v_0}{\partial x}+X=0\\ G\nabla^2 v+(\lambda+G)\dfrac{\partial\varepsilon v_0}{\partial y}+Y=0\end{cases}$$

b- Cas de la Contrainte Plane

$$\begin{cases}\varepsilon_x=\dfrac{1}{E}\left[\sigma_x-v\sigma_y\right]\\ \varepsilon_y=\dfrac{1}{E}\left[\sigma_y-v\sigma_x\right]\\ \gamma_{xy}=\dfrac{\tau_{xy}}{G}\end{cases}$$

$$\begin{cases}G\nabla^2 u+G\dfrac{(1+v)}{(1-v)}\dfrac{\partial\varepsilon_v}{\partial x}+X=0\\ G\nabla^2 v+G\dfrac{(1+v)}{(1-v)}\dfrac{\partial\varepsilon_v}{\partial y}+Y=0\end{cases}$$

$\varepsilon_v=\varepsilon_x+\varepsilon_y+\varepsilon_z$

4- EQUATIONS DE COMPATIBILITE OU EQUATION DE LEVY

a- Cas de Contrainte plane

$\dfrac{\partial\sigma_x}{\partial x}+\dfrac{\partial\tau_{xy}}{\partial y}+X=0$ (Dérive / x)

$\dfrac{\partial\tau_{xy}}{\partial x}+\dfrac{\partial\sigma_y}{\partial y}+Y=0$ (Dérive / y)

(Dérive /y)

$$\Longrightarrow \begin{cases} \frac{\partial^2 \sigma_x}{\partial_x{}^2} + \frac{\partial^2 \tau_{xy}}{\partial_x \partial_y} + \frac{\partial_X}{\partial_x} = 0 \\ \frac{\partial^2 \tau_{xy}}{\partial_x \partial_y} + \frac{\partial^2 \sigma_y}{\partial_y{}^2} + \frac{\partial_Y}{\partial_y} = 0 \end{cases}$$ Par sommassions des deux équations

$$\frac{\partial^2 \sigma_x}{\partial_x{}^2} + \frac{\partial^2 \sigma_y}{\partial_y{}^2} + 2\frac{\partial^2 \tau_{xy}}{\partial_x \partial_y} + \frac{\partial_X}{\partial_x} + \frac{\partial_Y}{\partial_y} = 0$$

$$2\frac{\partial^2 \tau_{xy}}{\partial_x \partial_y} = -\left[\frac{\partial^2 \sigma_x}{\partial_x{}^2} + \frac{\partial^2 \sigma_y}{\partial_y{}^2} + \frac{\partial_X}{\partial_x} + \frac{\partial_Y}{\partial_y}\right]$$

Équations compatibilité

$$\frac{\partial^2 \varepsilon_x}{\partial_y{}^2} + \frac{\partial^2 \varepsilon_y}{\partial_x{}^2} = \frac{\partial^2 \gamma_{xy}}{\partial_x \partial_y}$$

$\varepsilon_x = \frac{1}{E}\left[\sigma_x - v\sigma_y\right]$; $\varepsilon_y = \frac{1}{E}\left[\sigma_y - v\sigma_x\right]$; $\gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{2(1+v)}{E}\tau_{xy}$.

$$\frac{1}{E}\frac{\partial^2 \sigma_x}{\partial_y{}^2} - \frac{v}{E}\frac{\partial^2 \sigma_y}{\partial_y{}^2} + \frac{1}{E}\frac{\partial^2 \sigma_y}{\partial_x{}^2} - \frac{v}{E}\frac{\partial^2 \sigma_x}{\partial_x{}^2} = \frac{2(1+v)}{E}\frac{\partial^2 \tau_{xy}}{\partial_x \partial_y}$$

$$\frac{\partial^2 \sigma_x}{\partial_y{}^2} - v\frac{\partial^2 \sigma_y}{\partial_y{}^2} + \frac{\partial^2 \sigma_y}{\partial_x{}^2} - v\frac{\partial^2 \sigma_x}{\partial_x{}^2} = 2(1+v)\frac{\partial^2 \tau_{xy}}{\partial_x \partial_y}$$

$$\frac{\partial^2 \sigma_x}{\partial_y{}^2} - v\frac{\partial^2 \sigma_y}{\partial_y{}^2} + \frac{\partial^2 \sigma_y}{\partial_x{}^2} - v\frac{\partial^2 \sigma_x}{\partial_x{}^2} = -(1+v)\left[\frac{\partial^2 \sigma_x}{\partial_x{}^2} + \frac{\partial^2 \sigma_y}{\partial_y{}^2} + \frac{\partial_X}{\partial_x} + \frac{\partial_Y}{\partial_y}\right]$$

$$\frac{\partial^2 \sigma_x}{\partial_y{}^2} + \frac{\partial^2 \sigma_y}{\partial_x{}^2} = -\frac{\partial^2 \sigma_y}{\partial_y{}^2} - \frac{\partial^2 \sigma_x}{\partial_x{}^2} - (1+v)\left[\frac{\partial_X}{\partial_x} + \frac{\partial_Y}{\partial_y}\right]$$

$$\boxed{\nabla^2(\sigma_x + \sigma_y) = -(1+v)\left[\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right]}$$

b- Cas de déformation plane

$$\varepsilon_x = \frac{1}{E}\left[(1+v^2)\sigma_x - v(1+v)\sigma_y\right]$$

$$\varepsilon_y = \frac{1}{E}\left[(1+v^2)\sigma_y - v(1+v)\sigma_x\right]$$

$$\gamma_{xy} = \frac{2(1+v)}{E}\tau_{xy}$$

$$\boxed{\nabla^2(\sigma_x + \sigma_y) = -\frac{1}{(1-v)}\left[\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right]}$$

5- FONCTION CONTRAINTE ou FONCTION D'AIRY :

La solution d'un problème d'élasticité plan conduit à l'intégration des équations différentielles d'équilibre, de l'équation de compatibilité et à la vérification des conditions aux limites.

Soit $X = pg_x$ force gravitationnelle

$Y = pg_y$ p : masse volumique du corps
g_x et g_y : les composantes suivant x et y de l'accélération gravitationnelle
Les équations d'équilibre :

$$\frac{\partial\sigma_x}{\partial_x} + \frac{\partial\tau_{xy}}{\partial_y} + pg_x = 0$$

$$\frac{\partial\tau_{xy}}{\partial_x} + \frac{\partial\sigma_y}{\partial_y} + pg_y = 0$$

Equation de compatibilité

$$\nabla^2(\sigma_x + \sigma_y) = 0$$

Pour résoudre ce problème, Airy a introduit une fonction $\varphi\,(x.y)$
Dite fonction de contraintes tell que :

$$\sigma_x = \frac{\partial^2\varphi}{\partial_y{}^2}\,;\,\sigma_y = \frac{\partial^2\varphi}{\partial_x{}^2}\,;\,\tau_{xy} = -\frac{\partial^2\varphi}{\partial_y{}^2} - pg_x y - pg_y x \quad (*)$$

Si X et Y sont négligées

$$\Longrightarrow \sigma_x = \frac{\partial^2\varphi}{\partial_y{}^2}\,;\,\sigma_y = \frac{\partial^2\varphi}{\partial_x{}^2}\,;\,\tau_{xy} = -\frac{\partial^2\varphi}{\partial_x\partial_y} \quad (**)$$

Pour vérifier cette solution ; il faut faire la vérification de l'équation de compatibilité.

$$\frac{\partial^4\varphi}{\partial_x{}^4} + 2\frac{\partial^4\varphi}{\partial_x{}^2\partial_y{}^2} + \frac{\partial^4\varphi}{\partial_y{}^4} = 0$$

Où bien :

$$\boldsymbol{\nabla^4(\varphi) = 0}$$

Il reste donc à déterminer la fonction $\varphi\,(x.y)$
Les composantes de contraintes déterminées à partir de cette fonction en utilisant les équations (*) (**) doivent satisfaire aux conditions aux limites :

$$\bar{X} + l\sigma_x + m\tau_{xy}$$

$$\bar{Y} + l\tau_{xy} + m\sigma_y$$

6- <u>REALISATION DES PROBLEMES PLANS PAR LES PLOYNOMES :</u>

Le choix de la résolution des problèmes plans par les polynômes doit obligatoirement satisfaire aux conditions de la fonction d'Airy.

<u>a /*Cas d'1 polynôme de degré 2</u>

$$\varphi_2(x,y) = \frac{a_2}{2}x^2 + b_2xy + \frac{c_2}{2}y^2$$

$$\frac{\partial^4\varphi}{\partial_x{}^4} = 0\,;\,\frac{\partial^4\varphi}{\partial_x{}^2\partial_y{}^2} = 0\,;\,\frac{\partial^4\varphi}{\partial_y{}^4} = 0$$

Et donc

$$\sigma_x = \frac{\partial^2\varphi}{\partial_y{}^2} = c_2\,;\,\sigma_y = \frac{\partial^2\varphi}{\partial_x{}^2} = a_2\,;\,\tau_{xy} = -b_2 = \frac{\partial^2\varphi}{\partial_x\partial_y}$$

<u>b-*Cas d'1 polynôme de degré 3</u>

$$\varphi_3(x,y) = \frac{a_3}{6}x^3 + \frac{b_3}{2}xy^2 + \frac{c_3}{2}xy^2 + \frac{d_3}{6}y^3$$

$$\nabla^4\varphi_3(x,y) = 0\,;\,\frac{\partial^4\varphi}{\partial_x{}^4} = 0\,;\,\frac{\partial^4\varphi}{\partial_x{}^2\partial_y{}^2} = 0\,;\,\frac{\partial^4\varphi}{\partial_y{}^4} = 0$$

$$\begin{cases} \sigma_x = \dfrac{\partial^2 \varphi}{{\partial_y}^2} = c_3 x + d_3 y \\ \sigma_y = \dfrac{\partial^2 \varphi}{{\partial_x}^2} = a_3 x + b_3 y \\ \tau_{xy} = -\dfrac{\partial^2 \varphi}{\partial_x \partial_y} = -(b_3 x + c_3 y) \end{cases}$$

c/ *Cas d'1 polynôme de degré 4

$$\varphi_4(x,y) = \frac{a_4}{4*3}x^4 + \frac{b_4}{3*2}x^3 y + \frac{c_4}{2}x^2 y^2 + \frac{d_4}{3*2}x y^3 + \frac{e_4}{4*3}y^4$$

$$\nabla^2(\varphi) = \frac{\partial^4 \varphi_4}{{\partial_x}^4} + 2\frac{\partial^4 \varphi_4}{{\partial_x}^2 {\partial_y}^2} + \frac{\partial^4 \varphi_4}{{\partial_y}^4} = 0$$

$\frac{\partial^4 \varphi_4}{{\partial_x}^4} = 2a_4$; $\frac{\partial^4 \varphi_4}{{\partial_y}^4} = 2e_4$ Et $\frac{\partial^4 \varphi_4}{{\partial_x}^2 {\partial_y}^2} = 2e_4$

Il faut que $2a_4 + 2e_4 + 4c_4 = 0$

$e_4 = -(2c_4 + a_4)$

$$\sigma_x = \frac{\partial^2 \varphi_4}{{\partial_y}^2} = c_4 x^2 + d_4 xy - (2c_4 + a_4)y^2$$

$$\sigma_y = \frac{\partial^2 \varphi_4}{{\partial_x}^2} = a_4 x^2 + b_4 xy + c_4 y^2$$

$$\tau_{xy} = -\frac{\partial^2 \varphi_4}{\partial_x \partial_y} = -\frac{b_4}{2}x^2 - 2c_4 xy - \frac{d_4}{2}y^2$$

d/ * Cas d'un polynôme du degré 5

$$\varphi_5(x,y) = \frac{a_5}{5*4}x^5 + \frac{b_5}{4*3}x^4 y + \frac{c_5}{3*2}x^3 y^2 + \frac{d_5}{3*2}x^2 y^3 + \frac{e_5}{4*3}x y^4 + \frac{f_5}{5*4}y^5$$

$\nabla^4 \varphi_5 = 0$

$\frac{\partial^4 \varphi_5}{{\partial_x}^4} = 6a_5 x + 2b_5 y$; $\frac{\partial^4 \varphi_5}{{\partial_y}^4} = 2e_5 x + 6f_5 y$

$$\frac{\partial^4 \varphi_5}{{\partial_x}^2 {\partial_y}^2} = 2c_5 x + 2d_5 y$$

$\nabla^4 \varphi_5 = 6a_5 x + 2b_5 y + 2e_5 x + 6f_5 y + 4c_5 x + 4d_5 y = 0$

$(6a_5 + 2e_5 + 4c_5) + (2b_5 + 6f_5 + 4d_5)y = 0$

Donc il faut que :

$$\boxed{\begin{array}{c} e_5 = -(3a_5 + 2c_5) \\ f_5 = -1/3(b_5 + 2d_5) \end{array}}$$

$$\sigma_x = \frac{c_5}{3}x^3 + d_5 x^2 y + e_5 x y^2 + f_5 y^3$$

$$\sigma_y = a_5 x^3 + b_5 x^2 y + c_5 x y^2 + \frac{d_5}{3}y^3$$

$$\tau_{xy} = -\frac{b_5}{3}x^3 - c_5 x^2 y - d_5 x y^2 - \frac{e_5}{3}y^3$$